Azerbaijan Journal of Mathematics
V. 16, No 2, 2026, July
ISSN 2218-6816
https://doi.org/10.59849/2218-6816.2026.2.269

A Necessary and Sufficient Criterion for the Global Minimum of Multiextremal Functions

Zh.S. Tutkusheva*, Kh.T. Otarov

Abstract. Global optimization of multiextremal functions of several variables, i.e., functions possessing multiple local extrema, remains one of the most challenging problems in computational mathematics. The importance of this problem stems from its widespread occurrence in both theoretical research and practical applications.
In previous work by one of the authors, a special transformation of the original objective function was proposed to construct an auxiliary function. Several key properties of this auxiliary function, including non-negativity, uniform continuity, differentiability, and monotonicity, were rigorously established. These properties ensure computational efficiency and provide theoretical guarantees of convergence for the proposed global optimization method.
In this paper, necessary and sufficient conditions for the existence of a global minimum of the objective function are derived and proved. As a result, the original optimization problem is reduced to finding the greatest zero of a convex univariate function. It is shown that the global minimum of the objective function coincides with the supremum of the set of zeros of the auxiliary function. This characterization enables the use of well-established numerical methods to compute the greatest zero with high accuracy.
Key Words and Phrases: objective function, global optimization, auxiliary function, global minimum, necessary and sufficient conditions.
2010 Mathematics Subject Classifications: 90C26, 65K10

---

1. Introduction

A substantial body of research has been dedicated to the methods and challenges of global optimization [2, 4, 10, 12, 14]. Numerous algorithms and techniques have been developed to address global extremum problems for specific classes of objective functions. Rational approaches to global optimization continue to be actively explored across various domains of science and engineering

*Corresponding author.

[1, 3, 5, 7, 8, 13, 15, 22], driven by the ongoing demand for novel methods and the lack of a universal algorithm suitable for all problem types.

The main difficulties in global optimization typically stem from the multiextremal structure, high dimensionality, and non-convexity of the objective functions. These factors are further compounded by high computational costs, sensitivity to initial conditions, and other complicating aspects. Consequently, there remains a significant need for the development of more efficient and computationally economical methods capable of ensuring reliable convergence to the global extremum. A considerable number of studies have been devoted to addressing these challenges [11, 23, 24].

The scientific and technological significance of this research lies in the ubiquity of optimization problems involving the determination of maximum or minimum values in fields such as science, engineering, and economics. Effective solutions to such problems require the development of robust and theoretically justified optimization techniques.

In response to the limitations of existing methods-namely, their high computational cost and inability to guarantee rapid convergence to the global minimum-a new approach was proposed in [6]. This approach introduces a special function $g_m(F, \alpha)$, constructed based on the original objective function $F(x)$ defined on a given set E. It has been shown that this auxiliary function possesses several properties that enable the transformation of the original global optimization problem into the problem of finding the greatest zero of a specially constructed univariate function.

The objective of this paper is to rigorously formulate and prove necessary and sufficient conditions for the global minimum of a multiextremal function of several variables. These conditions are expressed in terms of the greatest zero of a convex function of a single variable.

2. Problem statement

Let $E \subset \mathbb{R}^n$ be a closed set in n-dimensional Euclidean space, and let $F : E \to \mathbb{R}$ be a continuous function. Hereafter, $F(x)$ will be referred to as the objective function. By the continuity of $F(x)$, the sublevel set

$$E(F, \alpha) = \{x \in E \mid F(x) \leq \alpha,\ \alpha \in \mathbb{R}\},$$

is measurable for every $\alpha \in \mathbb{R}$.

Given the objective function

$$F(x) = F(x_1, x_2, \ldots, x_n),$$

the goal is to find the global minimum of $F(x)$ over the domain E with a prescribed accuracy ε:

$$\hat{\alpha} = \operatorname*{glob\,min}_{x\in E} F(x).$$

In this study, the search for the global minimum is based on a special function $g_m(F,\alpha)$, where $\alpha \in \mathbb{R}$, which is constructed via a transformation of the original objective function:

$$g_m(F,\alpha) = \int_E \left[\, |F(x)-\alpha| - (F(x)-\alpha)\,\right]^m dx, \qquad m \in \mathbb{N},\ m > 1. \tag{1}$$

Here, $F(x) = F(x_1, x_2, \ldots, x_n)$ is a multivariate function defined on the domain E. Since $g_m(F,\alpha)$ is a function of a single variable α, the global optimization problem is thereby reduced from a multivariate to a univariate one. Notably, regardless of the number of local extrema of $F(x)$, the auxiliary function $g_m(F,\alpha)$ is convex. Furthermore, its rate of growth increases with the parameter m. The choice of the parameter m significantly affects the efficiency of locating the greatest zero of the function $g_m(F,\alpha)$, as discussed in [20].

Definition 1. *The function $g_m(F,\alpha)$ is referred to as the auxiliary function.*

3. Conditions for a global extremum

First, we establish a necessary condition for a global minimum of the objective function.

Theorem 1. *If $\min_{x\in E} F(x) = \hat{\alpha}$, then*

$$g_m(F,\hat{\alpha}) = 0. \tag{2}$$

Proof. Assume that the desired value of the global minimum is $\hat{\alpha}$. Then, for any $x \in E$, we have $F(x) \geq \hat{\alpha}$. According to the results of Lemma 1 [21], which proves that $g_m(C,\alpha) = 0$ for $\forall C \geq \alpha$, we obtain

$$\begin{aligned} g_m(F,\hat{\alpha}) &= \int_E \left[\, |F(x)-\hat{\alpha}| - (F(x)-\hat{\alpha})\,\right]^m d\mu = \\ &= \int_E \left[\, (F(x)-\hat{\alpha}) - (F(x)-\hat{\alpha})\,\right]^m d\mu = 0. \end{aligned}$$

◄

Corollary 1. *The global minimum of the objective function $F(x)$ is a zero of the auxiliary function $g_m(F,\alpha)$.*

A sufficient condition for the global minimum is established and justified in the following theorem.

Theorem 2. *If* $\max_{\alpha\in\mathbb{R}}\{\alpha \in \mathbb{R} : g_m(F,\alpha) = 0\,\} = \hat{\alpha}$, *then* $\operatorname{glob}\min_{x\in E} F(x) = \hat{\alpha}$.

Proof. Assume the contrary. Suppose that condition (2) holds and that $\tilde{\alpha}$ is the value of the global minimum, but $\tilde{\alpha} \neq \hat{\alpha}$. Let $2\beta = \hat{\alpha} - \tilde{\alpha}$, with $\beta > 0$. Then the set $E(F,\tilde{\alpha})$ is a subset of $E(F,\hat{\alpha})$; therefore,

$$\mu(E(F,\hat{\alpha})) > \mu(E(F,\tilde{\alpha})).$$

It follows that

$$\begin{aligned}
g_m(F,\hat{\alpha}) &= \int_E [\,|F(x)-\hat{\alpha}| - F(x) + \hat{\alpha}\,]^m \; d\mu = \\
&= \int_{E\setminus E(F,\hat{\alpha})} [\,|F(x)-\hat{\alpha}| - F(x) + \hat{\alpha}\,]^m \, d\mu + \\
&\quad + \int_{E(F,\hat{\alpha})} [\,|F(x)-\hat{\alpha}| - F(x) + \hat{\alpha}\,]^m \, d\mu = \\
&= \int_{E\setminus E(F,\hat{\alpha})} [\,(F(x)-\hat{\alpha}) - F(x) + \hat{\alpha}\,]^m \, d\mu + \\
&\quad + \int_{E(F,\hat{\alpha})} [\,-(F(x)-\hat{\alpha}) - F(x) + \hat{\alpha}\,]^m \, d\mu = \\
&= \int_{E(F,\hat{\alpha})} [\,2(\hat{\alpha} - F(x))\,]^m \, d\mu.
\end{aligned}$$

Then

$$g_m(F,\hat{\alpha}) = \int_{E(F,\hat{\alpha})} [\,2(\hat{\alpha} - F(x))\,]^m \, d\mu. \tag{3}$$

Since $\hat{\alpha} = \tilde{\alpha}+2\beta$ and $\hat{\alpha} > \tilde{\alpha}+\beta > \tilde{\alpha}$, then for the sets $E(F,\hat{\alpha}) = \{\,x \in E \mid F(x) \leq \hat{\alpha}\,\}$, $E(F,\tilde{\alpha}+\beta) = \{\,x \in E \mid F(x) \leq \tilde{\alpha}+\beta\,\}$, $E(F,\tilde{\alpha}) = \{\,x \in E \mid F(x) \leq \tilde{\alpha}\,\}$, the following relations hold:

$$E(F,\tilde{\alpha}) \subset E(F,\tilde{\alpha}+\beta) \subset E(F,\hat{\alpha}).$$

If $\tilde{\alpha} < F(x) \leq \tilde{\alpha}+\beta < \hat{\alpha}$, it is easy to verify that

$$\hat{\alpha} - F(x) > \hat{\alpha} - (\tilde{\alpha}+\beta) = \beta.$$

Transforming integral (3), we obtain the following lower bound:

$$\begin{aligned} g_m(F,\hat{\alpha}) &= \int_{E(F,\hat{\alpha})} [\,2(\hat{\alpha}-F(x))\,]^m d\mu = \\ &= \int_{E(F,\hat{\alpha})\setminus E(F,\tilde{\alpha}+\beta)} [\,2(\hat{\alpha}-F(x))\,]^m d\mu + \int_{E(F,\tilde{\alpha}+\beta)} [\,2(\hat{\alpha}-F(x))\,]^m d\mu \geq \\ &\geq \int_{E(F,\tilde{\alpha}+\beta)} [\,2(\hat{\alpha}-F(x))\,]^m d\mu > \\ &> [2\beta]^m \int_{E(F,\tilde{\alpha}+\beta)} d\mu = [2\beta]^m \mu(E(F,\tilde{\alpha}+\beta)) > 0. \end{aligned}$$

The last inequality contradicts equality (2). Therefore, the initial assumption is false, and the theorem is proven. ◄

Corollary 2. *The exact value of the greatest zero of the auxiliary function $g_m(F,\alpha)$ coincides with the exact value of the global minimum of the objective function $F(x)$.*

4. Examples

In this section, we present specific examples that clearly demonstrate the effectiveness of the proposed method.

Example 1. *To illustrate the theoretical results, we consider a test function and demonstrate how the objective function is transformed into the corresponding auxiliary function. As a first example, we use the well-known Ackley function, which is widely employed as a benchmark in global optimization:*

$$F(x,y) = -20e^{-0.2\sqrt{0.5(x^2+y^2)}} - e^{0.5(\cos(2\pi x)+\cos(2\pi y))} + e + 20. \tag{4}$$

This function is characterized by a large number of local minima in the domain:

$$-5 \leq x \leq 5, \quad -5 \leq y \leq 5.$$

The surface plot of the objective function $F(x,y)$ over the specified domain is shown in Figure 1. Its graph in this domain is depicted using MATLAB. The auxiliary function for the Ackley function for $m=6$ will look like:

$$\begin{aligned} g_6(F,\alpha) = \int_{-5}^{5}\int_{-5}^{5} \Big[& \Big|-20e^{-0.2\sqrt{0.5(x^2+y^2)}} - e^{0.5(\cos(2\pi x)+\cos(2\pi y))} + e + 20 - \alpha\Big| \\ & -\big(-20e^{-0.2\sqrt{0.5(x^2+y^2)}} - e^{0.5(\cos(2\pi x)+\cos(2\pi y))} + e + 20 - \alpha\big)\Big]^6 dx\,dy. \end{aligned} \tag{5}$$

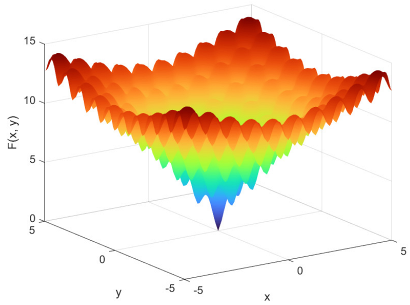

Figure 1: Graph of the Ackley function.

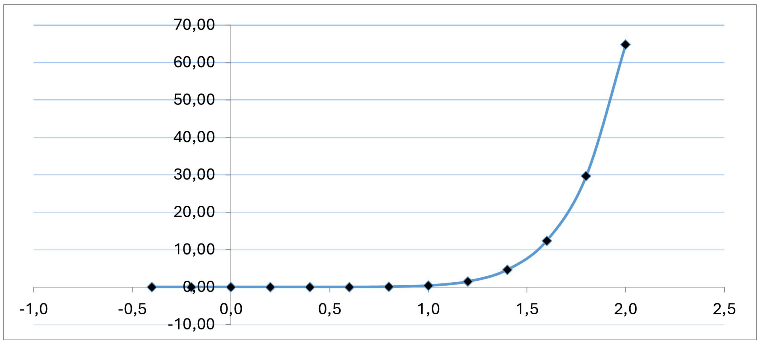

Figure 2: Graph of the auxiliary function (5).

The graph of function (5) is shown in Figure 2. The reference value of the global minimum of the Eckley function: $F(0;0) = 0$*. Computation result:* $F\left(10^{-10};\, 10^{-10}\right) = 10^{-10}$*. Method accuracy:* $\varepsilon = 10^{-10}$.

In the works of A. V. Kuznetsov and A. I. Ruban [9, 18], test functions were constructed using hyperbolic and exponential potential models. Building upon these methods, we generated a family of test functions exhibiting multiple local minima in a four-dimensional space. A key feature of these functions is the ability to explicitly predefine both the number and the coordinates of the local minima, providing a controlled environment for evaluating global optimization algorithms.

Example 2. *Consider a three-variable function constructed based on a hyperbolic potential that has four local minima in the cube* $[-10;10] \times [-10;10] \times [-10;10]$*:*

$$
\begin{aligned}
F(x,y,z) = &-\frac{1}{5\left(|x-3|^2+|y-4|^2+|z+1|^2\right)+1}-\\
&-\frac{1}{2\left(|x+5|^5+|y+8|^5+|z-2|^5\right)+1.8}-\\
&-\frac{1}{10\left(|x+5|^{2.2}+|y-6|^{2.2}+|z-9|^{2.2}\right)+3}-\\
&-\frac{1}{2\left(|x-7|^{2.5}+|y+8|^{2.5}+|z+1|^{2.5}\right)+2.5}.
\end{aligned}
\tag{6}
$$

The auxiliary function for the objective function (6) for $m = 6$ is given by:

$$
g_6(F,\alpha) = \int_{-10}^{10}\int_{-10}^{10}\int_{-10}^{10} \left[\,|F(x,y,z)-\alpha| - (F(x,y,z)-\alpha)\right]^6 \, dx\,dy\,dz. \tag{7}
$$

Function (7) is a strictly increasing function. It is illustrated graphically in Figure 3. According to the newly proposed method, the greatest zero of the aux-

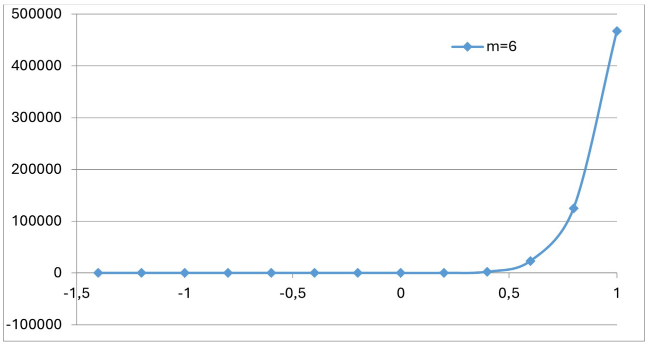

Figure 3: Graph of the auxiliary function (7).

iliary function was found after 42 iterations with an accuracy of $\varepsilon = 10^{-10}$: $\hat{\alpha} \approx -1.00132552058$. The algorithm for finding the largest zero of the function was implemented in the Visual Studio software environment.

5. Conclusion

In this paper, the problem of finding the global minimum of a multivariable, multi-extremal objective function has been reformulated as the problem of determining the greatest zero of an auxiliary function that depends on a single scalar parameter. This auxiliary function exhibits several advantageous analytical properties, including non-negativity, strict convexity, and uniform continuity.

Consequently, the original high-dimensional, non-convex optimization problem is effectively transformed into a one-dimensional convex optimization task.

The computation of the auxiliary function $g_m(F, \alpha)$, represented as a multiple integral over the domain E, is carried out using cubature formulas-specifically, the Sobolev cubature formulas [19]. This research direction was further developed through the works of M.D. Ramazanov, who made substantial contributions to the theory and application of multidimensional integration in this context [16, 17].

Numerical integration and evaluation of the auxiliary function were implemented in the C++ programming environment using custom-developed software tools. To determine the greatest zero of the auxiliary function with high accuracy, various classical and modern numerical root-finding algorithms can be applied or adapted, including the dichotomy method, the golden section method, Newton's method (method of tangents), gradient descent, and the Wegstein method.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgment

This research was funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP25794821).

References

[1] Floudas, C.A., Akrotirianakis, I.G., Caratzoulas, S., Meyer, C.A., Kallrath, J. (2005) Global optimization in the 21st century: Advances and challenges, Computers & Chemical Engineering, **29**, 1185–1202.

[2] Floudas, C.A., Pardalos, P.M. (2014) Recent Advances in Global Optimization, Princeton University Press, Princeton.

[3] González-Díaz, J., González-Rodríguez, B., Leal, M., Puerto, J. (2021) Global optimization for bi-level portfolio design: Economic insights from the Dow Jones Index, Omega, **102**, 1–18.

[4] Horst, R., Pardalos, P.M., Thoai, N.V. (2000) Introduction to Global Optimization, Springer, New York.

[5] Huster, W.R., Bongartz, D., Mitsos, A. (2017) Deterministic global optimization of the design of a geothermal organic Rankine cycle, Energy Procedia, **129**, 50–57.

[6] Kaidasov, Zh., Tutkusheva, Zh. (2021) Algorithm for calculating the global minimum of a smooth function of several variables, Mathematical Modelling of Engineering Problems, **8(4)**, 591–596.

[7] Klepeis, J.L., Pieja, M.J., Floudas, C.A. (2003) Hybrid global optimization algorithms for protein structure prediction, Biophysical Journal, **84(2)**, 869–882.

[8] Kunde, C., Michaels, D., Micovic, J., Lutze, P., Górak, A., Kienle, A. (2015) Deterministic global optimization in conceptual process design of distillation and melt crystallization, Chemical Engineering and Processing: Process Intensification, **99**, 132–142.

[9] Kuznetsov, A.V., Ruban, A.I. (2010) Search for the main minima of multiextremal functions with active consideration of inequality constraints, Journal of Siberian Federal University. Engineering & Technologies, **3**, 335–346.

[10] Liberti, L., Maculan, N. (eds.) (2006) Global Optimization: From Theory to Implementation, Springer, New York.

[11] Liberti, L., Sergei, K. (2005) Comparison of deterministic and stochastic approaches to global optimization, International Transactions in Operational Research, **12(3)**, 263–285.

[12] Locatelli, M., Schoen, F. (eds.) (2013) Global Optimization: Theory, Algorithms, and Applications, SIAM, Philadelphia.

[13] Locatelli, M., Schoen, F. (2021) (Global) optimization: Historical notes and recent developments, EURO Journal on Computational Optimization, **9(1)**, 1–15.

[14] Numerica, A. (2004) Complete Search in Continuous Global Optimization and Constraint Satisfaction, Cambridge University Press, Cambridge.

[15] Pinter, J.D. (2006) Global Optimization: Scientific and Engineering Case Studies, Springer, New York.

[16] Ramazanov, M.D. (2010) Theory of lattice cubature formulas with a bounded boundary layer, Ufa Mathematical Journal, **2(3)**, 63–82.

[17] Ramazanov, M.D. (2013) Asymptotically optimal lattice cubature formulas with a bounded boundary layer and the unsaturation property, Matematicheskii Sbornik, **204(7)**, 71–96.

[18] Ruban, A.I. (2013) Global optimization method based on the selective averaging coordinate with restrictions, Journal of Control and Computer Science, **1(22)**, 114–123.

[19] Sobolev, S.L., Vaskevich, V.L. (1996) Cubature Formulas, Sobolev Mathematical University, Novosibirsk.

[20] Tutkusheva, Zh., Otarov, Kh.T. (2024) Application of the auxiliary function method to the search for the global minimum of functions of many variables, Mathematical Modelling of Engineering Problems, **118(5)**, 1323–1329.

[21] Tutkusheva, Zh.S., Kazbekova, G.N., Seilkhanova, R.B., Kairakbaev, A.K. (2022) Wegstein's method for calculating the global extremum, Mathematical Modelling of Engineering Problems, **9(2)**, 405–410.

[22] Wu, Z. (1996) The effective energy transformation scheme as a special continuation approach to global optimization with application to molecular conformation, SIAM Journal on Optimization, **6(3)**, 748–768.

[23] Zhigljavsky, A., Zilinskas, A. (2007) Stochastic Global Optimization, Springer, New York.

[24] Zhu, W., Ali, M.M. (2009) Solving nonlinearly constrained global optimization problems via an auxiliary function method, Journal of Computational and Applied Mathematics, **230(2)**, 491–503.

Zhailan S. Tutkusheva
Department of Mathematics, Physical and Mathematical Faculty, K. Zhubanov Aktobe Regional Univerity, Aktobe, Kazakhstan
E-mail: zhailan_k@mail.ru

Khassen T. Otarov
Department of Mathematics, Physical and Mathematical Faculty, K. Zhubanov Aktobe Regional Univerity, Aktobe, Kazakhstan
E-mail: khassenotar@mail.ru

Received 04 November 2025
Accepted 16 June 2026