Azerbaijan Journal of Mathematics
V. 16, No 1, 2026, January
ISSN 2218-6816
https://doi.org/10.59849/2218-6816.2026.1.116

Nonlinear Anisotropic Parabolic Problem Involving a Singular Nonlinearity

H. Abdelaziz*, R. Mecheter

Abstract. In this work, we prove the existence and uniqueness of both entropy solution and renormalized solution for an anisotropic singular parabolic $\overrightarrow{q}$-Laplacian equations using the penalization method. Moreover, we prove that the entropy solution coincide with the renormalized solution.
Key Words and Phrases: anisotropic singular parabolic equations, anisotropic Sobolev spaces, penalization method, entropy solutions, renormalized solutions, existence, uniqueness.

2010 Mathematics Subject Classifications: 35K55, 35K67, 46E35

---

1. Introduction

This paper investigates a nonlinear Dirichlet parabolic problem. We begin by considering the model problem:

$$\begin{cases} \partial_t u - \Delta u + \gamma u = \dfrac{f}{u^\theta} & \text{in } Q_T, \\ u(x,t) = 0 & \text{on } \Sigma_T, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is an open bounded domain with Lipschitz boundary $\partial\Omega$. Here, $Q_T = \Omega \times (0,T)$ denotes the space-time cylinder with $T > 0$, and $\Sigma_T = \partial\Omega \times (0,T)$ represents the lateral boundary. We assume that $\gamma > 0$, $f \in L^1(Q_T)$ is non-negative, and $0 < \theta \leq 1$. The initial condition $u_0 \in L^1(\Omega)$ is strictly positive on compact subsets of Ω, i.e., for every $\omega \subset\subset \Omega$, there exists $C_\omega > 0$ such that $u_0(x) \geq C_\omega > 0$ in ω.

We extend problem (1) to the more general anisotropic case:

*Corresponding author.

$$\begin{cases} \partial_t u - \Delta_{\overrightarrow{q}}(u) + \gamma(x,t)|u|^{q_0-2}u = \dfrac{f}{u^\theta} & \text{in } Q_T, \\ u(x,t) = 0 & \text{on } \Sigma_T, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{2}$$

where $\overrightarrow{q} = (q_0, q_1, \dots, q_N)$ with $1 < q_i < \infty$ for $i = 0, 1, \dots, N$. The operator $\Delta_{\overrightarrow{q}}$ is a Leray-Lions type operator defined by

$$\Delta_{\overrightarrow{q}}(u) := \sum_{i=1}^{N} \partial_{x_i}\left(|D^i u|^{q_i-2} D^i u\right), \quad \text{where } D^i u = \partial_{x_i} u,$$

a mapping between the anisotropic parabolic space $L^{\overrightarrow{q}}(0,T; W^{1,\overrightarrow{q}}(\Omega))$ and its dual $L^{\overrightarrow{q}'}(0,T; W^{-1,\overrightarrow{q}'}(\Omega))$. The coefficient $\gamma(x,t) \in L^\infty(Q_T)$ satisfies $\gamma(x,t) \geq \gamma_0 > 0$ almost everywhere in Q_T.

Problem (2) arises in various physical contexts, including chemical heterogeneous catalyst kinetics [4], thermo-conductivity problems [14], boundary layer phenomena for viscous fluids [15], electromagnetic field theory [16], non-Newtonian fluid dynamics [22], and turbulent flow of gas in porous media [23]. The study of nonlinear parabolic equations has led to two fundamental solution concepts: renormalized solutions, introduced by Di Perna and Lions [11] for the Boltzmann equation, and entropy solutions developed independently by Bénilan et al. [6]. Subsequent developments in entropy solutions can be found in [3].

For the isotropic nonsingular case ($\theta = 0$), Boccardo et al. [7] established existence and regularity results for solutions of

$$\begin{cases} \partial_t u - \operatorname{div}(|\nabla u|^{q-2}\nabla u) + \gamma_0 |u|^{q_0-1}u = f & \text{in } Q_T, \\ u(x,t) = 0 & \text{on } \Sigma_T, \\ u(x,0) = 0 & \text{in } \Omega, \end{cases}$$

with $q > 1 + \frac{N}{N+1}$, $q_0 > \frac{p(N+1)-N}{N}$, $\gamma_0 > 0$, and $f \in L^1(Q_T)$.

In the anisotropic setting ($\theta = 0$), Mokhtari et al. [20] considered measure-valued data:

$$\begin{cases} \partial_t u + Au + F(t,x,u,Du) = f & \text{in } Q_T, \\ u(x,t) = 0 & \text{on } \Sigma_T, \\ u(x,0) = f_0 & \text{in } \Omega, \end{cases}$$

where $Au = -\operatorname{div}(a(x,t,u,Du))$ and the Carathéodory functions a and F satisfy appropriate anisotropic growth conditions.

For constant exponent anisotropic problems, Chrif et al. [8] proved existence results for entropy and renormalized solutions of

$$\begin{cases} \partial_t u + Au + F(t,x,u) = f & \text{in } Q_T, \\ u(x,t) = 0 & \text{on } \Sigma_T, \\ u(x,0) = u_0 & \text{in } \Omega, \end{cases}$$

where $f \in L^1(Q_T)$, $u_0 \in L^1(\Omega)$, $Au = -\text{div}(a(x,t,u,Du))$, and the Carathéodory functions $a(x,t,u,\xi)$ and $F(x,t,u)$ satisfy some anisotropic growth conditions.

The variable exponent case was studied by Mecheter et al. [19]:

$$\begin{cases} \partial_t u - \sum_{i=1}^N D_i(d_i(t,x,u)a_i(t,x,Du)) + F(t,x,u) = f & \text{in } \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0 & \text{on } \mathbb{R}^N, \end{cases}$$

where $T > 0$, $f \in L^1(0,T; L^1_{\text{loc}}(\mathbb{R}^N))$, $u_0 \in L^1_{\text{loc}}(\mathbb{R}^N)$, and the Carathéodory functions a_i, d_i, and F satisfy anisotropic growth conditions.

Moreover, Abdelaziz et al. [2] proved the existence and regularity for

$$\begin{cases} \partial_t u - \displaystyle\sum_{i=1}^N \left[D_i \left(\frac{|D_i u|^{q_i(x)-2} D_i u}{\big(\ln(e+|u|)\big)^{\sigma(x)}} \right) + |u|^{s_i(x)} \right] = f & \text{in } Q_T, \\ u(x,t) = 0 & \text{on } \Sigma_T, \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $q_i(\cdot), s_i(\cdot) \in C(\overline{\Omega})$ take values in $(1,\infty)$, $m, \sigma \in C(\overline{\Omega})$ with $m(\cdot) > 1$ and $\sigma(\cdot) \geq 0$, $f \in L^{m(\cdot)}(Q_T)$, and $u_0 \in L^{(m(\cdot)-1)s_+(\cdot)+1}(\Omega)$ with $s_+(\cdot) = \max_{1\leq i\leq N} s_i(\cdot)$.

For singular isotropic problems ($\theta \neq 0$), De Bonis et al. [9] established existence and regularity results for

$$\begin{cases} \partial_t u - \Delta_q u = \dfrac{f}{u^\theta} & \text{in } Q_T, \\ u(x,t) = 0 & \text{on } \Sigma_T, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

with $\theta > 0$, $q \geq 2$, $f > 0$, $f \in L^m(Q_T)$ $(m \geq 1)$, and $u_0 \in L^\infty(\Omega)$.

Khelifi et al. [18] considered more general nonlinearities:

$$\begin{cases} \partial_t u - \text{div}(a(x,t,u,Du)) = fg(u) & \text{in } Q_T, \\ u(x,t) = 0 & \text{on } \Sigma_T, \\ u(x,0) = 0 & \text{in } \Omega, \end{cases}$$

where g satisfies $g(s) \le C/s^\theta$ $(0 < \theta < 1)$ and a satisfies

$$a(x,t,u,Du)\cdot Du \ge \frac{\alpha|Du|^q}{(1+|u|)^\sigma}, \quad 0 \le \sigma < q-1+\frac{q}{N}+\theta\left(1+\frac{q}{N}\right),$$

with $\alpha > 0$.

Additionally, Mounim et al. [21] addressed

$$\begin{cases} \partial_t u - \operatorname{div}(a(x,t,u,Du)) + |u|^{s-1}u = fg(u) & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \\ u(x,0) = 0 & \text{in } \Omega, \end{cases}$$

where g satisfies $g(s) \le C/s^\theta$ $(0 < \theta < 1)$ and a satisfies

$$a(x,t,u,Du)\cdot Du \ge \frac{\alpha|Du|^q}{(1+|u|)^{\sigma(q-1)}}, \quad 0 \le \sigma < 1,$$

with $\alpha > 0$ and $C > 0$.

Recently, Zaater et al. [26] studied the existence of bounded solutions for

$$\begin{cases} \partial_t u - \sum_{i=1}^{N} D_i\left(\frac{u^{q_i-1}(1+|Du|)^{-1}Du + |Du|^{q_i-2}Du}{(1+|u|)^\sigma}\right) = \frac{f}{u^\theta} & \text{in } Q_T, \\ u(x,t) = 0 & \text{on } \Sigma_T, \\ u(x,0) = 0 & \text{in } \Omega, \end{cases}$$

where $2 \le q_i < N$ for every $i = 1,\dots,N$, $\sigma,\theta \ge 0$, and $0 \le f \in L^m(Q_T)$ with $m > \frac{N}{\overline{q}} + 1$. Their approach was based on Stampacchia's lemma, using carefully chosen test functions to obtain a priori estimates.

Regarding parabolic problems involving $q(x)$-Laplacian, a power, and a singular nonlinearity, we mention the result of Panda et al. [24], where the authors proved the existence of a non-negative weak solution for

$$\begin{cases} \partial_t u - \Delta_{q(x)} u = \lambda u^{p(x)-1} + u^{-\delta(x)} g + f & \text{in } Q_T, \\ u(x,t) = 0 & \text{on } \Sigma_T, \\ u(x,0) = u_0 & \text{in } \Omega, \end{cases}$$

where $\lambda \in (0,\infty)$, $f \in L^1(Q_T)$, $g \in L^\infty(\Omega)$, $u_0 \in L^r(\Omega)$ with $r \ge 2$, $\delta : \overline{\Omega} \to (0,\infty)$ is continuous, and $p, q \in C(\overline{\Omega})$ with $\max_{x\in\overline{\Omega}} q(x) < N$ and $p(\cdot) < q^*(\cdot)$. Two cases were distinguished according to the choice of f with different ranges of parameters $q(\cdot)$ and $p(\cdot)$.

Our work establishes the well-posedness (existence and uniqueness) of entropy and renormalized solutions to problem (2) using penalization methods. The main challenges consist of proving the convergence of gradients and handling the nonlinear dependence of the operator $-\Delta_{\overrightarrow{q}}(u)$ on the gradient Du. Moreover, the standard tools and techniques available for the nonsingular setting cannot be directly extended to the singular setting.

The paper is organized as follows: Section 2 presents preliminary results on anisotropic parabolic spaces. Section 3 proves the existence of entropy solutions, and the final section establishes their equivalence with renormalized solutions.

2. Preliminaries on Anisotropic Function Spaces

In this section, we present fundamental definitions and results concerning anisotropic Sobolev spaces and parabolic spaces used to study our parabolic $\overrightarrow{q}$-Laplacian problem (2). Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be an open bounded domain with boundary $\partial\Omega$. Consider $N+1$ exponents $q_0, q_1, \ldots, q_N$ satisfying $1 < q_i < \infty$ for $i = 0, \ldots, N$. We define: $\overrightarrow{q} = (q_0, q_1, \ldots, q_N)$, $D^0 u = u$, and $D^i u = \partial_{x_i} u$, $q_- = \min\{q_0, q_1, \ldots, q_N\} > 1$

2.1. Anisotropic Sobolev Spaces

The anisotropic Sobolev space is defined as

$$W^{1,\overrightarrow{q}}(\Omega) = \left\{u \in L^{q_0}(\Omega) \mid D^i u \in L^{q_i}(\Omega) \text{ for } i = 1, \ldots, N\right\},$$

equipped with the norm

$$\|u\|_{W^{1,\overrightarrow{q}}(\Omega)} = \sum_{i=0}^{N} \|D^i u\|_{L^{q_i}(\Omega)}. \tag{3}$$

The space $W_0^{1,\overrightarrow{q}}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,\overrightarrow{q}}(\Omega)$ under the norm (3). Both $W^{1,\overrightarrow{q}}(\Omega)$ and $W_0^{1,\overrightarrow{q}}(\Omega)$ are separable and reflexive Banach spaces.

Proposition 1 ([13]). *For any $u \in W_0^{1,\overrightarrow{q}}(\Omega)$, the following statements hold:*

(i) ***Poincaré inequality:*** *There exists $C_q > 0$ such that*

$$\|u\|_{L^{q_i}(\Omega)} \leq C_q \|D^i u\|_{L^{q_i}(\Omega)} \quad \textit{for } i = 1, \ldots, N.$$

(ii) ***Sobolev inequality:*** *There exists $C_s > 0$ such that*

$$\|u\|_{L^r(\Omega)} \le \frac{C_s}{N} \sum_{i=1}^{N} \|D^i u\|_{L^{q_i}(\Omega)},$$

where $\frac{1}{\overline{q}} = \frac{1}{N} \sum_{i=1}^N \frac{1}{q_i}$ *and* $r = \begin{cases} \overline{q}^* = \frac{N\overline{q}}{N-\overline{q}} & \text{if } \overline{q} < N, \\ \text{any } r \in [1, +\infty) & \text{if } \overline{q} \ge N. \end{cases}$

Proposition 2 ([10])**.** *Let $\Omega \subset \mathbb{R}^N$ ($N \ge 2$) be bounded. Then:*

- *If $\overline{q} < N$, then $W_0^{1,\overrightarrow{q}}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$ is compact for any $r \in [1, q_-^*)$, where $\frac{1}{q_-^*} = \frac{1}{q_-} - \frac{1}{N}$.*
- *If $\overline{q} = N$, then $W_0^{1,\overrightarrow{q}}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$ is compact for any $r \in [1, +\infty)$.*
- *If $\overline{q} > N$, then $W_0^{1,\overrightarrow{q}}(\Omega) \hookrightarrow\hookrightarrow L^\infty(\Omega) \cap C^0(\overline{\Omega})$ is compact.*

Proposition 3 ([5])**.** *The dual space $W^{-1,\overrightarrow{q'}}(\Omega)$ of $W_0^{1,\overrightarrow{q}}(\Omega)$, where $\overrightarrow{q'} = (q'_0, q'_1, \ldots, q'_N)$ with $\frac{1}{q'_i} + \frac{1}{q_i} = 1$, has the following properties:*

- *For each $F \in W^{-1,\overrightarrow{q'}}(\Omega)$, there exist $f_0 \in L^{q'_0}(\Omega)$ and $f_i \in L^{q'_i}(\Omega)$ ($i = 1, \ldots, N$) such that $F = f_0 - \sum_{i=1}^N D^i f_i$.*
- *The duality pairing is given by $\langle F, u \rangle = \sum_{i=0}^N \int_\Omega f_i D^i u \, dx \quad \forall u \in W_0^{1,\overrightarrow{q}}(\Omega)$.*
- *The dual norm is $\|F\|_{W^{-1,\overrightarrow{q'}}(\Omega)} = \inf \left\{ \sum_{i=0}^N \|f_i\|_{L^{q'_i}(\Omega)} \mid F = f_0 - \sum_{i=1}^N D^i f_i \right\}$.*

For complete proofs and additional details, we refer to [5, 13].

2.2. Anisotropic Parabolic Spaces

Let $Q_T = \Omega \times (0, T)$ with $0 < T < \infty$. We define the anisotropic parabolic space:

Definition 1. *The space $L^{\overrightarrow{q}}(0, T; W^{1,\overrightarrow{q}}(\Omega))$ consists of all measurable functions $u\colon Q_T \to \mathbb{R}$ satisfying $\sum_{i=0}^N \int_0^T \|D^i u(t)\|_{L^{q_i}(\Omega)}^{q_i} dt < \infty$ and equipped with the norm $\|u\|_{L^{\overrightarrow{q}}(0,T;W^{1,\overrightarrow{q}}(\Omega))} = \sum_{i=0}^N \|D^i u\|_{L^{q_i}(Q_T)}$.*

Definition 2. *The space $L^{\overrightarrow{q}}(0, T; W_0^{1,\overrightarrow{q}}(\Omega))$ is defined as*

$$L^{\overrightarrow{q}}(0, T; W_0^{1,\overrightarrow{q}}(\Omega)) = \left\{ u \in L^{\overrightarrow{q}}(0, T; W^{1,\overrightarrow{q}}(\Omega)) \mid u = 0 \text{ on } \partial\Omega \times [0, T] \right\}.$$

Remark 1. *Both* $L^{\overrightarrow{q}}(0,T;W^{1,\overrightarrow{q}}(\Omega))$ *and* $L^{\overrightarrow{q}}(0,T;W_0^{1,\overrightarrow{q}}(\Omega))$ *are separable and reflexive Banach spaces.*

Definition 3. *Let* $k > 0$. *The truncation function* $T_k\colon \mathbb{R} \to \mathbb{R}$ *is defined by* $T_k(s) = \begin{cases} s & \text{if } |s| \le k, \\ k\,\mathrm{sgn}(s) & \text{if } |s| > k, \end{cases}$ *where* sgn *denotes the sign function. Its primitive* $\varphi_k\colon \mathbb{R} \to \mathbb{R}^+$ *is given by* $\varphi_k(r) = \int_0^r T_k(s)ds = \begin{cases} \frac{r^2}{2} & \text{if } |r| \le k, \\ k|r| - \frac{k^2}{2} & \text{if } |r| > k. \end{cases}$ *Note that* $\varphi_k(r) \ge 0$ *and* $\varphi_k(r) \le k|r|$ *for all* $r \in \mathbb{R}$.

We define the space

$$\mathcal{T}_0^{1,\overrightarrow{q}}(Q_T) = \\ = \big\{u\colon Q_T \to \mathbb{R} \text{ measurable} \;\big|\; T_k(u) \in L^{\overrightarrow{q}}(0,T;W_0^{1,\overrightarrow{q}}(\Omega)),\ D^i T_k(u) \in L^{q_i}(Q_T)\big\}.$$

Definition 4. *The dual space of* $L^{\overrightarrow{q}}(0,T;W_0^{1,\overrightarrow{q}}(\Omega))$ *is defined as*

$$L^{\overrightarrow{q'}}(0,T;W^{-1,\overrightarrow{q'}}(\Omega)) \\ = \left\{F = f_0 - \sum_{i=1}^N D^i f_i \;\middle|\; f_0 \in L^{q_0'}(Q_T),\ f_i \in L^{q_i'}(Q_T) \text{ for } i = 1,\dots,N\right\},$$

equipped with the norm

$$\|F\| = \inf\left\{\sum_{i=0}^N \|f_i\|_{L^{q_i'}(Q_T)} \;\middle|\; F = f_0 - \sum_{i=1}^N D^i f_i \text{ with } f_0 \in L^{q_0'}(Q_T), f_i \in L^{q_i'}(Q_T)\right\}.$$

The duality pairing between $L^{\overrightarrow{q}}(0,T;W^{1,\overrightarrow{q}}(\Omega))$ *and* $L^{\overrightarrow{q'}}(0,T;W^{-1,\overrightarrow{q'}}(\Omega))$ *is given by*

$$\int_0^T \langle F, v\rangle dt = \sum_{i=0}^N \int_{Q_T} f_i D^i v \,dx\,dt \quad \text{for all } v \in L^{\overrightarrow{q}}(0,T;W_0^{1,\overrightarrow{q}}(\Omega)).$$

Proposition 4 ([27])**.** *Let* $u \in \mathcal{T}_0^{1,\overrightarrow{q}}(Q_T)$. *For each* $i = 1,\dots,N$, *there exists a unique measurable function* $\nu_i\colon Q_T \to \mathbb{R}$ *such that*

$$D^i T_k(u) = \nu_i \cdot \chi_{\{|u|<k\}} \quad \text{a.e. in } Q_T \text{ for all } k > 0,$$

where χ_A *denotes the characteristic function of* A.

The functions ν_i *are called the weak partial derivatives of* u *and are denoted by* $D^i u$. *Moreover, if* $u \in L^1(0,T;W_0^{1,1}(\Omega))$, *then* ν_i *coincides with the standard distributional derivative of* u, *i.e.,* $\nu_i = D^i u$.

Proposition 5 ([25]). *Let q and p satisfy either $1 \le q < \infty$ and $p = 1$, or $q = \infty$ and $p > 1$. Consider three Banach spaces E_1, F, and E_2 such that the embedding $E_1 \hookrightarrow\hookrightarrow F$ is compact, and the embedding $F \hookrightarrow E_2$ is continuous.*

If $(u_n)_n$ is a bounded sequence in $L^q(0,T;E_1)$ with $\left(\frac{\partial u_n}{\partial t}\right)_n$ bounded in $L^p(0,T;E_2)$, then there exists $u \in L^q(0,T;F)$ such that, for some subsequence,

$$u_n \to u \quad \text{strongly in } L^q(0,T;F).$$

That is, $(u_n)_n$ is precompact in $L^q(0,T;F)$.

Remark 2. *For the case $q = p = 1$, take $E_1 = W_0^{1,1}(\Omega)$, $F = L^1(\Omega)$, and $E_2 = W^{-1,1}(\Omega)$. Here we have the compact embedding $W_0^{1,1}(\Omega) \hookrightarrow\hookrightarrow L^1(\Omega)$, and the continuous embedding $L^1(\Omega) \hookrightarrow W^{-1,1}(\Omega)$.*

This gives the compact embedding

$$\left\{ u \in L^1(0,T;W_0^{1,1}(\Omega)) \,\Big|\, \frac{\partial u}{\partial t} \in L^1(0,T;W^{-1,1}(\Omega)) \right\} \hookrightarrow\hookrightarrow L^1(Q_T).$$

Furthermore, when $q_- > 1$, the continuous embeddings

$$W_0^{1,\overrightarrow{q}}(\Omega) \hookrightarrow W_0^{1,1}(\Omega) \quad \text{and} \quad W^{-1,\overrightarrow{q}'}(\Omega) \hookrightarrow W^{-1,1}(\Omega)$$

imply the compactness of

$$\left\{ u \in L^{\overrightarrow{q}}(0,T;W_0^{1,\overrightarrow{q}}(\Omega)) \,\Big|\, \frac{\partial u}{\partial t} \in L^{\overrightarrow{q}'}(0,T;W^{-1,\overrightarrow{q}'}(\Omega)) \right\} \hookrightarrow\hookrightarrow L^1(Q_T). \tag{4}$$

Proposition 6. *([17]) Let $(u_n)_n$ be a sequence in $L^1(\Omega)$ and $u \in L^1(\Omega)$ be such that $u_n \to u$ almost everywhere in Ω, $u_n \ge 0$ and $u \ge 0$ almost everywhere in Ω, and $\int_\Omega u_n\,dx \to \int_\Omega u\,dx$. Then $u_n \to u$ strongly in $L^1(\Omega)$.*

Proposition 7. *Let $(u_n)_n$ be a sequence in $L^{\overrightarrow{q}}(0,T;W_0^{1,\overrightarrow{q}}(\Omega))$ such that $(\partial_t u_n)_n$ is bounded in $L^{\overrightarrow{q}'}(0,T;W^{-1,\overrightarrow{q}'}(\Omega))$, $u_n \rightharpoonup u$ weakly in $L^{\overrightarrow{q}}(0,T;W_0^{1,\overrightarrow{q}}(\Omega))$, and the following convergence holds:*

$$\begin{aligned} &\sum_{i=1}^{N} \int_{Q_T} \left(|D^i u_n|^{q_i-2} D^i u_n - |D^i u|^{q_i-2} D^i u\right)(D^i u_n - D^i u)\,dx\,dt \\ &\quad + \int_{Q_T} \left(|u_n|^{q_0-2} u_n - |u|^{q_0-2} u\right)(u_n - u)\,dx\,dt \to 0. \end{aligned} \tag{5}$$

Then $u_n \to u$ strongly in $L^{\overrightarrow{q}}(0,T;W_0^{1,\overrightarrow{q}}(\Omega))$ for a subsequence.

Proof. Following the approach in [8], by using (5) and Proposition 6 we get $|u_n|^{q_i} \to |u|^{q_i}$ in $L^1(Q_T)$, and $|D^i u_n|^{q_i} \to |D^i u|^{q_i}$ in $L^1(Q_T)$, which establishes $u_n \to u$ in $L^{\overrightarrow{q}}(0,T;W_0^{1,\overrightarrow{q}}(\Omega))$. ◄

3. Existence Results

3.1. Entropy Solutions

We begin by defining the concept of an entropy solution to problem (2).

Definition 5. *For $0 < \theta \leq 1$, $0 \leq u_0 \in L^1(\Omega)$, and $0 \leq f \in L^1(Q_T)$, a function $u \in \mathcal{T}_0^{1,\overrightarrow{q}}(Q_T) \cap C([0,T]; L^1(\Omega))$ is called an* entropy solution *to problem* (2) *if it satisfies:*

1. ***Positivity condition**: u is strictly positive on $\Omega \times (0,T)$, meaning that for every $\omega \subset\subset \Omega$, there exists $C_\omega > 0$ such that: $u \geq C_\omega > 0$ in $\omega \times (0,T)$, and $|u|^{q_0-2}u \in L^1(Q_T)$.*

2. ***Entropy inequality**: For all test functions $\phi \in L^{\overrightarrow{q}}(0,T; W_0^{1,\overrightarrow{q}}(\Omega)) \cap L^\infty(Q_T)$ with $\partial_t \phi \in L^{\overrightarrow{q}'}(0,T; W^{-1,\overrightarrow{q}'}(\Omega)) + L^1(Q_T)$, the following holds:*

$$\int_\Omega \varphi_k(u-\phi)(T)dx - \int_\Omega \varphi_k(u-\phi)(0)dx + \int_{Q_T} \partial_t \phi \cdot T_k(u-\phi)dxdt$$
$$+ \sum_{i=1}^N \int_{Q_T} |D^i u|^{q_i-2} D^i u \cdot D^i T_k(u-\phi)dxdt + \int_{Q_T} \gamma(x,t)|u|^{q_0-2}u \cdot T_k(u-\phi)dxdt$$
$$\leq \int_{Q_T} \frac{f}{u^\theta} T_k(u-\phi)dxdt,$$

where φ_k is the primitive of the truncation function T_k.

Theorem 1. *Under the assumptions $0 < \theta \leq 1$, $0 \leq f \in L^1(Q_T)$, and $0 \leq u_0 \in L^1(\Omega)$, there exists at least one entropy solution to problem* (2)*.*

Proof. **Claim 1: Non-singular Approximate Problem**
Consider sequences $(f_n)_n$ in $L^{\overrightarrow{p}'}(0,T; W^{-1,\overrightarrow{p}'}(\Omega)) \cap L^1(Q_T)$ and $(u_{0,n})_n$ in $\mathcal{C}_0^\infty(\Omega)$ such that $f_n \to f$ in $L^1(Q_T)$ with $f_n = T_n(f)$ and $0 \leq f_n \leq f$, and $u_{0,n} \to u_0$ in $L^1(\Omega)$ with $0 \leq u_{0,n} = T_n(u_0) \leq u_0$.

We study the approximate problem

$$\begin{cases} \frac{\partial u_n}{\partial t} + \Phi_n u_n = \dfrac{f_n}{\left(|T_n(u_n)| + \frac{1}{n}\right)^\theta} & \text{in } Q_T, \\ u_n(x,t) = 0 & \text{on } \Sigma_T, \\ u_n(x,0) = u_0 & \text{in } \Omega, \end{cases} \tag{6}$$

where the operator Φ_n is defined by $\Phi_n u := -\Delta_{\overrightarrow{q}} u + \gamma(x,t)|u|^{q_0-2}u$, with $\gamma \in L^\infty(Q_T)$ satisfying $\gamma(x,t) \geq \gamma_0 > 0$ a.e. in Q_T. ◄

Remark 3. *For a fixed $n \in \mathbb{N}$, the right-hand side of Problem 6 is bounded in $L^\infty(Q_T)$. This allows us to employ standard methods (such as Schauder's fixed-point theorem [1, 12], variational methods [27], or the semi-discretization approach [24]) to establish the existence of solutions to Problem 6.*

The following lemma establishes the existence of solutions to the approximate Problem 6.

Lemma 1. *For a fixed $n \in \mathbb{N}$, Problem 6 admits a unique non-negative weak solution $u_n \in L^{\overrightarrow{q}}(0,T;W_0^{1,\overrightarrow{q}}(\Omega))$.*

Proof. Consider the operator $\Phi_n : L^{\overrightarrow{q}}(0,T;W_0^{1,\overrightarrow{q}}(\Omega)) \longrightarrow L^{\overrightarrow{q}'}(0,T;W^{-1,\overrightarrow{q}'}(\Omega))$, which is pseudo-monotone, bounded, and coercive in the following sense:

$$\frac{\int_0^T \langle \Phi_n v, v\rangle \, dt}{\|v\|_{L^{\overrightarrow{q}}(0,T;W_0^{1,\overrightarrow{q}}(\Omega))}} \to +\infty \quad \text{as} \quad \|v\|_{L^{\overrightarrow{q}}(0,T;W_0^{1,\overrightarrow{q}}(\Omega))} \to +\infty.$$

Step 1: Pseudo-monotonicity of Φ_n

Let $(u_k)_{k\geq 1}$ be a sequence in $L^{\overrightarrow{q}}(0,T;W_0^{1,\overrightarrow{q}}(\Omega))$ such that

$$\begin{cases} u_k \rightharpoonup u & \text{in } L^{\overrightarrow{q}}(0,T;W_0^{1,\overrightarrow{q}}(\Omega)),\\ \Phi_n u_k \rightharpoonup \chi_n & \text{in } L^{\overrightarrow{q}'}(0,T;W^{-1,\overrightarrow{q}'}(\Omega)),\\ \limsup_{k\to+\infty}\langle \Phi_n u_k, u_k\rangle \leq \langle \chi_n, u\rangle. \end{cases} \tag{7}$$

We aim to show that $\chi_n = \Phi_n u$ and $\langle \Phi_n u_k, u_k\rangle \to \langle \chi_n, u\rangle$ as $k \to +\infty$.

From $u_k \rightharpoonup u$ in $L^{\overrightarrow{q}}(0,T;W_0^{1,\overrightarrow{q}}(\Omega))$ and the compact embedding (4) we deduce that $u_k \to u$ strongly in $L^1(Q_T)$ and almost everywhere in Q_T.

Since $(u_k)_k$ is bounded in $L^{\overrightarrow{q}}(0,T;W_0^{1,\overrightarrow{q}}(\Omega))$, the growth condition of $(|\xi|^{q_i-2}\xi)$ implies that $(|D^i u_k|^{q_i-2}D^i u_k)_k$ is bounded in $L^{q_i'}(Q_T)$. Thus, there exists $\eta_i \in L^{q_i'}(Q_T)$ such that

$$|D^i u_k|^{q_i-2}D^i u_k \rightharpoonup \eta_i \quad \text{in } L^{q_i'}(Q_T), \quad \text{for all } i = 1,\dots,N. \tag{8}$$

Moreover,

$$|u_k|^{q_0-2}u_k \rightharpoonup |u|^{q_0-2}u \quad \text{in } L^{q_0'}(Q_T). \tag{9}$$

For any $v \in L^{\overrightarrow{q}}(0,T;W_0^{1,\overrightarrow{q}}(\Omega))$, we have

$$\langle \chi_n, v\rangle = \sum_{i=1}^N \int_{Q_T} \eta_i D^i v \, dx\, dt + \int_{Q_T} \gamma(x,t)|u|^{q_0-2}uv \, dx\, dt. \tag{10}$$

Combining (7) and (10), we obtain

$$\limsup_{k\to+\infty}\langle \Phi_n u_k, u_k\rangle \le \sum_{i=1}^{N}\int_{Q_T}\eta_i D^i u\,dx\,dt + \int_{Q_T}\gamma(x,t)|u|^{q_0}\,dx\,dt.$$

Thus,

$$\begin{aligned}\limsup_{k\to+\infty}&\left\{\sum_{i=1}^{N}\int_{Q_T}|D^i u_k|^{q_i}\,dx\,dt + \int_{Q_T}\gamma(x,t)|u_k|^{q_0}\,dx\,dt\right\}\\ &\le \sum_{i=1}^{N}\int_{Q_T}\eta_i D^i u\,dx\,dt + \int_{Q_T}\gamma(x,t)|u|^{q_0}\,dx\,dt.\end{aligned} \tag{11}$$

On the other hand, the monotonicity condition yields

$$\begin{aligned}&\sum_{i=1}^{N}\int_{Q_T}\left(|D^i u_k|^{q_i-2}D^i u_k - |D^i u|^{q_i-2}D^i u\right)\left(D^i u_k - D^i u\right)dx\,dt\\ &+\int_{Q_T}\gamma(x,t)\left(|u_k|^{q_0-2}u_k - |u|^{q_0-2}u\right)(u_k-u)\,dx\,dt \ge 0.\end{aligned}$$

Using (8) and (9) and combining the result with (11), we obtain

$$\begin{aligned}&\lim_{k\to+\infty}\left\{\sum_{i=1}^{N}\int_{Q_T}|D^i u_k|^{q_i}\,dx\,dt + \int_{Q_T}\gamma(x,t)|u_k|^{q_0}\,dx\,dt\right\}\\ &=\sum_{i=1}^{N}\int_{Q_T}\eta_i D^i u\,dx\,dt + \int_{Q_T}\gamma(x,t)|u|^{q_0}\,dx\,dt.\end{aligned} \tag{12}$$

From (10) and (12), it follows that $\langle \Phi_n u_k, u_k\rangle \to \langle \chi_n, u\rangle$ as $k\to+\infty$. Moreover, (12) and Proposition 7 gives us $u_k \to u$ in $L^{\overrightarrow{q}}(0,T;W_0^{1,\overrightarrow{q}}(\Omega))$. Consequently, $D^i u_k \to D^i u$ almost everywhere in Q_T, and $|D^i u_k|^{q_i-2}D^i u_k \to |D^i u|^{q_i-2}D^i u$ a.e. in Q_T. Thus, $|D^i u_k|^{q_i-2}D^i u_k \rightharpoonup |D^i u|^{q_i-2}D^i u$ in $L^{q_i'}(Q_T)$ for all $i=1,\dots,N$. Together with (9), this yields $\chi_n = \Phi_n u$.

Step 2: Boundedness of Φ_n

Using Hölder's inequality, we establish for all $u, v \in L^{\overrightarrow{q}}(0,T;W_0^{1,\overrightarrow{q}}(\Omega))$

$$\left|\int_0^T \langle \Phi_n u, v\rangle\,dt\right| \le C_0\|v\|_{L^{\overrightarrow{q}}(0,T;W_0^{1,\overrightarrow{q}}(\Omega))}.$$

This proves that Φ_n is bounded.

Step 3: Coercivity of Φ_n
By the growth condition, we have for all $u \in L^{\overrightarrow{q}}(0,T;W_0^{1,\overrightarrow{q}}(\Omega))$

$$\left|\int_0^T \langle \Phi_n u, u\rangle \, dt\right| \geq C_\alpha \|u\|^{q_-}_{L^{\overrightarrow{q}}(0,T;W_0^{1,\overrightarrow{q}}(\Omega))} - N - \gamma_0,$$

where $C_\alpha = \frac{\min\{1,\gamma_0\}}{(N+1)^{q_- - 1}}$. This proves the coercivity of Φ_n.

Step 4: Uniqueness
Suppose there exist two weak solutions u_n and v_n to Problem (6). Then $z_n = u_n - v_n$ satisfies

$$\begin{cases} \frac{\partial z_n}{\partial t} + \Phi_n z_n = \dfrac{f_n}{\left(|T_n(z_n)| + \frac{1}{n}\right)^\theta} & \text{in } Q_T, \\ z_n(x,t) = 0 & \text{on } \Sigma_T, \\ z_n(x,0) = u_0 & \text{in } \Omega. \end{cases}$$

Taking z_n as a test function, we conclude that $u_n = v_n$ almost everywhere in Q_T. ◄

The following lemma establishes the strict positivity of the sequence (u_n) of solutions to the approximate Problem (6), which we will apply later in the next claims.

Lemma 2. *Let u_n be a solution to Problem* (6) *given by Lemma 1. Then, for every $\omega \subset\subset \Omega$, there exists a constant $C_\omega > 0$ (independent of n) such that*

$$u_n \geq C_\omega > 0 \quad \text{in } \omega \times (0,T), \quad \text{for all } n \in \mathbb{N}.$$

Proof. The proof follows with minor modifications from the techniques employed in [12] (see also [1]). ◄

Claim 2: Weak Convergence of Truncations
Taking $T_k(u_n)$ $(k \geq 1)$ as a test function in Problem (6) and using Lemma 2, we get the uniform bound

$$\|T_k(u_n)\|_{L^{\overrightarrow{q}}(0,T;W_0^{1,\overrightarrow{q}}(\Omega))} \leq C_1 k^{\frac{1}{q_-}} \quad \text{for all} \quad k \geq 1. \tag{13}$$

To establish the decay estimate for the level sets, we observe that for any $k \geq 1$, $k\,\text{meas}\{u_n > k\} \leq C_2 k^{\frac{1}{q_-}}$, which implies the measure estimate

$$\text{meas}\{u_n > k\} \leq C_2 \frac{1}{k^{1-\frac{1}{q_-}}} \to 0 \quad \text{as} \quad k \to +\infty. \tag{14}$$

For any $\nu > 0$, we decompose the measure as follows:

$$\begin{aligned}\text{meas}\{|u_n - u_m| > \nu\} \leq \text{meas}\{u_n > k\} + \text{meas}\{u_m > k\} \\ + \text{meas}\{|T_k(u_n) - T_k(u_m)| > \nu\}.\end{aligned} \tag{15}$$

From (14), for any $\varepsilon > 0$, there exists $k_0 = k_0(\varepsilon) > 0$ such that

$$\text{meas}\{u_n > k\} \leq \frac{\varepsilon}{3} \quad \text{and} \quad \text{meas}\{u_m > k\} \leq \frac{\varepsilon}{3} \quad \forall k \geq k_0. \tag{16}$$

Moreover, by (13), the sequence $(T_k(u_n))_n$ is bounded in $L^{\overrightarrow{q}}(0,T;W_0^{1,\overrightarrow{q}}(\Omega))$. Thus, there exists a subsequence (still denoted by $(T_k(u_n))_n$) and a function $\eta_k \in L^{\overrightarrow{q}}(0,T;W_0^{1,\overrightarrow{q}}(\Omega))$ such that $T_k(u_n) \rightharpoonup \eta_k$ in $L^{\overrightarrow{q}}(0,T;W_0^{1,\overrightarrow{q}}(\Omega))$ as $n \to +\infty$. The compact embedding (4) implies the strong convergence

$$T_k(u_n) \to \eta_k \quad \text{in} \quad L^1(Q_T) \quad \text{and} \quad \text{a.e. in} \quad Q_T.$$

Consequently, $(T_k(u_n))_n$ is a Cauchy sequence in measure, and for any $\nu, \varepsilon > 0$, there exists $n_0 = n_0(k, \nu, \varepsilon)$ such that

$$\text{meas}\{|T_k(u_n) - T_k(u_m)| > \nu\} \leq \frac{\varepsilon}{3} \quad \text{for all} \quad n, m \geq n_0. \tag{17}$$

Combining (15)–(17), we conclude that for any $\varepsilon, \nu > 0$, there exists $n_0 = n_0(\nu, \varepsilon)$ such that $\text{meas}\{|u_n - u_m| > \nu\} \leq \varepsilon$ for all $n, m \geq n_0$. This shows that $(u_n)_n$ is a Cauchy sequence in measure. Hence, there exists a subsequence (still denoted by $(u_n)_n$) converging almost everywhere to some measurable function u, $u_n \to u$ a.e. in Q_T. Furthermore, the weak convergence of the truncations holds: $T_k(u_n) \rightharpoonup T_k(u)$ in $L^{\overrightarrow{q}}(0,T;W_0^{1,\overrightarrow{q}}(\Omega))$. Finally, by the Lebesgue dominated convergence theorem, we obtain the strong convergence $T_k(u_n) \to T_k(u)$ in $L^{q_0}(Q_T)$.

Claim 3: A Priori Estimates

Let $h > 0$. Taking $T_{h+1}(u_n) - T_h(u_n)$ as a test function in Problem (6) and using Lemma 2, we find

$$\begin{aligned}&\alpha \sum_{i=1}^{N} \int_{\{h \leq u_n < h+1\}} |D^i u_n|^{q_i}\, dx\, dt + \gamma_0 \int_{\{h+1 \leq u_n\}} (u_n)^{q_0 - 1}\, dx\, dt \\ &\leq \int_{\{u_n \geq h\}} \frac{f}{C_\omega^\theta}\, dx\, dt + \int_\Omega \varphi_{h+1}(u_{0,n})\, dx - \int_\Omega \varphi_h(u_{0,n})\, dx.\end{aligned} \tag{18}$$

Applying the Lebesgue dominated convergence theorem to the right-hand side terms of (18), and noting that $u_0 \in L^1(\Omega)$ and $C_\omega^\theta > 0$, we obtain:

$$\int_{\{u_n \geq h\}} f\, dx\, dt \to 0 \quad \text{as} \quad h \to +\infty, \tag{19}$$

and

$$\int_{\Omega}\varphi_{h+1}(u_{0,n})\,dx-\int_{\Omega}\varphi_{h}(u_{0,n})\,dx\leq$$
$$\leq\int_{\{h\leq u_{0,n}<h+1\}}\frac{1}{2}\,dx+\int_{\{h+1\leq u_{0,n}\}}u_{0}\,dx\to 0\quad\text{as}\quad h\to+\infty. \tag{20}$$

Combining (18)–(20), we obtain the following decay estimates:

$$\sum_{i=1}^{N}\int_{\{h\leq u_n<h+1\}}|D^{i}u_{n}|^{q_i}\,dx\,dt\to 0\quad\text{as}\quad h\to+\infty, \tag{21}$$

and

$$\int_{\{h+1\leq u_n\}}|u_n|^{q_0-1}\,dx\,dt\to 0\quad\text{as}\quad h\to+\infty. \tag{22}$$

Claim 4: Equi-integrability of the Nonlinear Terms By Claim 2, we have $|u_n|^{q_0-2}u_n\to|u|^{q_0-2}u$ a.e. in Q_T. To apply Vitali's convergence theorem, we verify the uniform equi-integrability of $(|u_n|^{q_0-2}u_n)_n$. From the estimate (22) in Claim 3, for any $\eta>0$ there exists $h(\eta)>0$ such that $\int_{\{u_n\geq h(\eta)\}}(u_n)^{q_0-1}\,dx\,dt\leq\frac{\eta}{2}$ for all $n\in\mathbb{N}$. The pointwise convergence and equi-integrability conditions allow us to apply Vitali's convergence theorem, yielding the desired strong convergence:

$$|u_n|^{q_0-2}u_n\to|u|^{q_0-2}u\quad\text{strongly in } L^{1}(Q_T). \tag{23}$$

Claim 5: Weak Convergence of Time Derivatives

Let $S_h\in C^{2}(\mathbb{R})$ be an increasing truncation function satisfying $S_h(r)=r$ for $|r|\leq h$, $\operatorname{supp}(S'_h)\subset[-h-1,h+1]$, and $\operatorname{supp}(S''_h)\subset[-h-1,-h]\cup[h,h+1]$.

For any test function $v\in L^{\overrightarrow{q}}(0,T;W_0^{1,\overrightarrow{q}}(\Omega))\cap L^{\infty}(Q_T)$, we use $S'_h(u_n)v$ as a test function in Problem (6) to obtain the estimate

$$\begin{aligned}&\left|\int_0^T\left\langle\frac{\partial S_h(u_n)}{\partial t},v\right\rangle dt\right|\\&\leq\sum_{i=1}^{N}\int_{Q_T}\left||D^iu_n|^{q_i-2}D^iu_n\right|\left|S'_h(u_n)D^iv+S''_h(u_n)vD^iu_n\right|dxdt\\&+\|\gamma\|_{L^{\infty}(Q_T)}\int_{Q_T}(u_n)^{q_0-1}|S'_h(u_n)v|dxdt+\int_{Q_T}\frac{f_n}{C_\omega^{\theta}}|S'_h(u_n)v|dxdt.\end{aligned} \tag{24}$$

Estimate of the Diffusion Terms: For the first term in (24), we have

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\{|u_n|\leq h+1\}}|D^iu_n|^{q_i-1}\left(|S'_h(u_n)||D^iv|+|S''_h(u_n)||v||D^iu_n|\right)dxdt\\&\leq C_4\left(\|v\|_{L^{\overrightarrow{q}}(0,T;W_0^{1,\overrightarrow{q}}(\Omega))}+\|v\|_{L^{\infty}(Q_T)}\right).\end{aligned} \tag{25}$$

Estimate of the Lower Order Terms: The remaining terms in (24) satisfy

$$\begin{aligned}&\|\gamma\|_{L^\infty(Q_T)}\int_{Q_T}|u_n|^{q_0-1}|S_h'(u_n)v|dxdt+\int_{Q_T}\frac{f_n}{C_\omega^\theta}|S_h'(u_n)v|dxdt\\&\leq C_5\left(\|v\|_{L^{\vec{q}}(0,T;W_0^{1,\vec{q}}(\Omega))}+\|v\|_{L^\infty(Q_T)}\right).\end{aligned}\tag{26}$$

Uniform Bound on Time Derivatives: Combining (24)-(26) yields:

$$\left|\int_0^T\left\langle\frac{\partial S_h(u_n)}{\partial t},v\right\rangle dt\right|\leq C_6\left(\|v\|_{L^{\vec{q}}(0,T;W_0^{1,\vec{q}}(\Omega))}+\|v\|_{L^\infty(Q_T)}\right),$$

where C_6 is independent of n.

This establishes that the sequence $\left(\frac{\partial S_h(u_n)}{\partial t}\right)_n$ is bounded in the space

$$L^{\vec{q}'}(0,T;W^{-1,\vec{q}'}(\Omega))+L^1(Q_T),$$

and consequently,

$$\frac{\partial S_h(u_n)}{\partial t}\rightharpoonup\frac{\partial S_h(u)}{\partial t}\quad\text{in }L^{\vec{q}'}(0,T;W^{-1,\vec{q}'}(\Omega))+L^1(Q_T).\tag{27}$$

Claim 6: Strong Convergence of the Gradients

Let $k\leq h$ and consider sufficiently large n. Using $S_h'(u_n)(T_k(u_n)-T_k(u))$ as a test function in Problem (6), we obtain the decomposition

$$\mathcal{J}_{n,h}^1+\mathcal{J}_{n,h}^2+\mathcal{J}_{n,h}^3+\mathcal{J}_{n,h}^4=\mathcal{J}_{n,h}^5$$

where the terms are defined as

$$\begin{aligned}\mathcal{J}_{n,h}^1&=\int_0^T\int_\Omega\frac{\partial S_h(u_n)}{\partial t}(T_k(u_n)-T_k(u))dxdt,\\\mathcal{J}_{n,h}^2&=\sum_{i=1}^N\int_{Q_T}S_h'(u_n)(|D^iu_n|^{q_i-2}D^iu_n)(D^iT_k(u_n)-D^iT_k(u))dxdt,\\\mathcal{J}_{n,h}^3&=\sum_{i=1}^N\int_{Q_T}(T_k(u_n)-T_k(u))S_h''(u_n)|D^iu_n|^{q_i}dxdt,\\\mathcal{J}_{n,h}^4&=\int_{Q_T}\gamma(x,t)|u_n|^{q_0-2}u_nS_h'(u_n)(T_k(u_n)-T_k(u))dxdt,\\\mathcal{J}_{n,h}^5&=\int_{Q_T}\frac{f_nS_h'(u_n)(T_k(u_n)-T_k(u))}{(|T_n(u_n)|+\frac{1}{n})^\theta}dxdt.\end{aligned}$$

Analysis of $\mathcal{J}^1_{n,h}$: From the weak convergence in (27), we have:

$$\liminf_{n\to\infty} \mathcal{J}^1_{n,h} = \liminf_{n\to\infty} \int_\Omega \varphi_k(S_h(u_n(T)))dx - \int_\Omega \varphi_k(S_h(u(T)))dx \geq 0,$$

where the non-negativity follows from Fatou's lemma and the pointwise convergence of $\varphi_k(S_h(u_n(T)))$.
Analysis of $\mathcal{J}^2_{n,h}$: Using the properties of S_h and the weak convergence of gradients, we obtain

$$\mathcal{J}^2_{n,h} \geq \sum_{i=1}^N \int_{Q_T} \left(|D^iT_k(u_n)|^{q_i-2}D^iT_k(u_n) - |D^iT_k(u)|^{q_i-2}D^iT_k(u)\right) \times (D^iT_k(u_n) - D^iT_k(u))dxdt + \varepsilon_2(n),$$

where $\varepsilon_2(n) \to 0$ as $n \to \infty$ by the weak convergence properties.
Analysis of $\mathcal{J}^3_{n,h}$: Since $\mathrm{supp}(S''_h) \subset [-h-1,-h] \cup [h,h+1]$, we have

$$\mathcal{J}^3_{n,h} \leq C\sum_{i=1}^N \int_{\{h<u_n\leq h+1\}} |D^iT_{h+1}(u_n)|^{q_i}dxdt \to 0 \quad \text{as } n,h\to\infty.$$

Analysis of $\mathcal{J}^4_{n,h}$ and $\mathcal{J}^5_{n,h}$: The remaining terms satisfy

$$\mathcal{J}^4_{n,h} \geq \gamma_0 \int_{Q_T} \left(|T_k(u_n)|^{q_0-2}T_k(u_n) - |T_k(u)|^{q_0-2}T_k(u)\right)(T_k(u_n)-T_k(u))dxdt+\varepsilon_3(n).$$

and $\mathcal{J}^5_{n,h} \leq \frac{1}{(n+\frac{1}{n})^\theta} \int_{Q_T} f_n|T_k(u_n) - T_k(u)|dxdt \to 0 \quad \text{as } n\to\infty$

Combining all estimates yields:

$$\lim_{n\to\infty}\Bigg[\sum_{i=1}^N \int_{Q_T} \left(|D^iT_k(u_n)|^{q_i-2}D^iT_k(u_n) - |D^iT_k(u)|^{q_i-2}D^iT_k(u)\right) \times (D^iT_k(u_n) - D^iT_k(u))dxdt + \gamma_0 \int_{Q_T} \left(|T_k(u_n)|^{q_0-2}T_k(u_n) - |T_k(u)|^{q_0-2}T_k(u)\right)(T_k(u_n) - T_k(u))dxdt\Bigg] = 0$$

Applying Proposition 7, we conclude the strong convergence

$$T_k(u_n) \to T_k(u) \text{ in } L^{\overrightarrow{q}}(0,T;W_0^{1,\overrightarrow{q}}(\Omega)) \text{ for all } k>0,$$
$$\text{and } D^iu_n \to D^iu \text{ a.e. in } Q_T.$$

Claim 7: The Convergence of u_n in $C([0,T];L^1(\Omega))$

Let u_n (resp. u_m) be the weak solution of the approximate problem (6) for the integer n (resp. m), and for $0 < s \leq T$. Taking $\psi = T_1(u_n - u_m) \cdot \chi_{[0,s]}$ as a test function and using Young's inequality, (23), and $D^i T_1(u_n - u_m) = (D^i u_n - D^i u_m) \cdot \chi_{\{|u_n - u_m| \leq 1\}}$, we deduce that $\int_\Omega |u_n(s) - u_m(s)|\, dx \to 0$, as $n, m \to +\infty$. Hence, u_n is a Cauchy sequence in $C([0,T]; L^1(\Omega))$. Therefore, u_n converges to $u \in C([0,T]; L^1(\Omega))$ and for all $0 \leq s \leq T$ we have $u_n(s) \to u(s)$ in $L^1(\Omega)$.

Claim 8: Passage to the Limit

Let $\Psi \in L^{\overrightarrow{q}}(0,T; W_0^{1,\overrightarrow{q}}(\Omega)) \cap L^\infty(Q_T)$ with $\partial_t \Psi \in L^{\overrightarrow{q}'}(0,T; W^{-1,\overrightarrow{q}'}(\Omega)) + L^1(Q_T)$ and $M = k + \|\Psi\|_{L^\infty(Q_T)}$ with $k > 0$.

We take $T_k(u_n - \Psi)$ as a test function in (6). Then

$$\begin{aligned} &\int_0^T \Big\langle \frac{\partial u_n}{\partial t}, T_k(u_n - \Psi) \Big\rangle\, dt + \sum_{i=1}^N \int_{Q_T} |D^i u_n|^{q_i - 2} D^i u_n \times D^i T_k(u_n - \Psi)\, dx\, dt \\ &\quad + \int_{Q_T} \gamma(x,t) |u_n|^{q_0 - 2} u_n T_k(u_n - \Psi)\, dx\, dt \\ &\quad = \int_{Q_T} \frac{f_n}{\big(|T_n(u_n)| + \frac{1}{n}\big)^\theta} T_k(u_n - \Psi)\, dx\, dt. \end{aligned} \tag{28}$$

For the first term on the left-hand side of (28), since $u_n \to u$ in $C([0,T]; L^1(\Omega))$, we have $u_n(T) \to u(T)$ in $L^1(\Omega)$. It follows that

$$\begin{aligned} &\int_\Omega \varphi_k(u_{0,n} - \Psi(0))\, dx \to \int_\Omega \varphi_k(u_0 - \Psi(0))\, dx, \text{ as } n \to +\infty, \\ &\text{and } \int_\Omega \varphi_k(u_n(T) - \Psi(T))\, dx \to \int_\Omega \varphi_k(u(T) - \Psi(T))\, dx, \text{ as } n \to +\infty, \end{aligned} \tag{29}$$

Now, we have $\frac{\partial \Psi}{\partial t} \in L^{\overrightarrow{q}'}(0,T; W^{-1,\overrightarrow{q}'}(\Omega)) + L^1(Q_T)$, and since $T_k(u_n - \Psi) \rightharpoonup T_k(u - \Psi)$ in $L^{\overrightarrow{q}}(0,T; W_0^{1,\overrightarrow{q}}(\Omega))$ and weak-$\star$ in $L^\infty(Q_T)$, we get

$$\int_{Q_T} \frac{\partial \Psi}{\partial t} T_k(u_n - \Psi)\, dx\, dt \to \int_{Q_T} \frac{\partial \Psi}{\partial t} T_k(u - \Psi)\, dx\, dt. \tag{30}$$

Concerning the second term on the left-hand side of (28), using Fatou's Lemma, we obtain

$$\begin{aligned} &\liminf_{n \to +\infty} \sum_{i=1}^N \int_{Q_T} |D^i u_n|^{q_i - 2} D^i u_n \times D^i T_k(u_n - \Psi)\, dx\, dt \\ &\quad = \int_{Q_T} |D^i u|^{q_i - 2} D^i u \times D^i T_k(u - \Psi)\, dx\, dt. \end{aligned} \tag{31}$$

Also, since $T_k(u_n-\Psi)\rightharpoonup T_k(u-\Psi)$ weak-$\star$ in $L^\infty(Q_T)$, and thanks to (23), we deduce that

$$\begin{aligned}&\int_{Q_T}\gamma(x,t)|u_n|^{q_0-2}u_nT_k(u_n-\Psi)\,dx\,dt\\ &\qquad\longrightarrow\int_{Q_T}\gamma(x,t)|u|^{q_0-2}uT_k(u-\Psi)\,dx\,dt,\end{aligned}\tag{32}$$

and similar to the proof of [[26], Lemma 5.2] (see also [[12], (3.7)]) , we deduce

$$\int_{Q_T}\frac{f_n}{\left(|T_n(u_n)|+\frac{1}{n}\right)^\theta}T_k(u_n-\Psi)\,dx\,dt\to\int_{Q_T}\frac{f}{u^\theta}T_k(u-\Psi)\,dx\,dt,\tag{33}$$

By combining (28)-(33), we conclude the proof.

3.2. Renormalized solutions

Here, we give the definition of the concept of a renormalized solution to problem (2).

Definition 6. *For $0<\theta\le 1$, $0\le u_0\in L^1(\Omega)$, and $0\le f\in L^1(Q_T)$, a function $u\in\mathcal{T}_0^{1,\overrightarrow{q}}(Q_T)\cap C([0,T];L^1(\Omega))$ is called a* renormalized solution *to problem* (2) *if it satisfies:*

1. ***Positivity condition**: u is strictly positive on $\Omega\times(0,T)$, meaning that for every $\omega\subset\subset\Omega$, there exists $C_\omega>0$ such that: $u\ge C_\omega>0$ in $\omega\times(0,T)$, and $|u|^{q_0-2}u\in L^1(Q_T)$.*

2. ***Renormalized equality**: For all test functions $\phi\in L^{\overrightarrow{q}}(0,T;W_0^{1,\overrightarrow{q}}(\Omega))\cap L^\infty(Q_T)$ with $\partial_t\phi\in L^{\overrightarrow{q'}}(0,T;W^{-1,\overrightarrow{q'}}(\Omega))+L^1(Q_T)$, the following holds:*

   - $\displaystyle\lim_{h\to+\infty}\sum_{i=1}^N\int_{\{h\le u\le h+1\}}|D^iu|^{q_i}\,dx\,dt=0,$
   - $$\begin{aligned}&\int_{Q_T}\frac{\partial S(u)}{\partial t}\varphi\,dx\,dt+\sum_{i=1}^N\int_{Q_T}|D^iu|^{q_i-2}D^iu\cdot(S''(u)\varphi D^iu+S'(u)D^i\varphi)\,dx\,dt\\ &+\int_{Q_T}\gamma(x,t)|u|^{q_0-2}u\cdot S'(u)\varphi\,dx\,dt\\ &=\int_{Q_T}\frac{fS'(u)\varphi}{u^\theta}\,dx\,dt,\end{aligned}\tag{34}$$

for every function $\varphi \in L^{\overrightarrow{q}}(0,T;W_0^{1,\overrightarrow{q}}(\Omega)) \cap L^\infty(Q_T)$, *and any renormalization* $S(\cdot) \in C^\infty(\mathbb{R})$ *such that* $\operatorname{supp} S'(\cdot) \subseteq [-M, M]$ *for some constant* $M > 0$.

Theorem 2. *Under the assumptions* $0 < \theta \leq 1$, $0 \leq f \in L^1(Q_T)$, *and* $0 \leq u_0 \in L^1(\Omega)$, *the entropy solution* u *in Theorem 1 is also a renormalized solution for problem* (2).

Proof. We shall prove that every entropy solution u satisfies all the properties of renormalized solutions.

Indeed, in view of Theorem 1, there exists a subsequence $(u_n)_n$ of solutions for the approximate problems (6) such that $T_k(u_n)$ strongly converges to $T_k(u)$ in $L^{\overrightarrow{q}}(0,T;W_0^{1,\overrightarrow{q}}(\Omega))$ for any $k > 0$, and satisfies $|u_n|^{q_0-2}u_n \to |u|^{q_0-2}u$ strongly in $L^1(Q_T)$. Also, thanks to (21) and Fatou's Lemma, we deduce that

$$\begin{aligned}
&\sum_{i=1}^N \int_{\{h\leq u<h+1\}} |D^i u|^{q_i}\,dx\,dt \\
&\quad\leq \liminf_{h\to+\infty} \sum_{i=1}^N \int_{\{h\leq u_n<h+1\}} |D^i u_n|^{q_i}\,dx\,dt \to 0, \text{ as } h \to +\infty.
\end{aligned}$$

Now, we will show the equality (34). Let $\varphi \in L^{\overrightarrow{q}}(0,T;W_0^{1,\overrightarrow{q}}(\Omega)) \cap L^\infty(Q_T)$ and $S(\cdot) \in C^\infty(\mathbb{R})$ with $\operatorname{supp} S'(\cdot) \subset [-M, M]$ for some constant $M > 0$. By taking $S'(u_n)\varphi$ as a test function in (6), we obtain

$$\begin{aligned}
&\int_0^T \Big\langle \frac{\partial u_n}{\partial t}, S'(u_n)\varphi \Big\rangle\,dt \\
&\quad+ \sum_{i=1}^N \int_{Q_T} |D^i u_n|^{q_i-2} D^i u_n \times D^i(S'(u_n)\varphi)\,dx\,dt \\
&\quad+ \int_{Q_T} \gamma(x,t)|u_n|^{q_0-2}u_n S'(u_n)\varphi\,dx\,dt \\
&\quad= \int_{Q_T} \frac{f_n S'(u_n)\varphi}{\big(|T_n(u_n)| + \frac{1}{n}\big)^\theta}\,dx\,dt.
\end{aligned} \tag{35}$$

First, from (27), we have $\frac{\partial S_h(u_n)}{\partial t} \rightharpoonup \frac{\partial S_h(u)}{\partial t}$ in $L^{\overrightarrow{q}'}(0,T;W^{-1,\overrightarrow{q}'}(\Omega)) + L^1(Q_T)$, and then

$$\lim_{n\to+\infty} \int_0^T \Big\langle \frac{\partial u_n}{\partial t}, S'(u_n)\varphi \Big\rangle\,dt = \int_{Q_T} \frac{\partial S(u)}{\partial t}\varphi\,dx\,dt. \tag{36}$$

Concerning the second term on the left-hand side of (35), we have

$$\int_{Q_T} |D^i u_n|^{q_i-2} D^i u_n \cdot D^i(S'(u_n)\varphi)\,dx\,dt$$

$$= \int_{Q_T} |D^i T_M(u_n)|^{q_i-2} D^i T_M(u_n) \cdot (S''(u_n)\varphi D^i T_M(u_n) + S'(u_n) D^i \varphi)\, dx\, dt.$$

We have $(|D^i T_M(u_n)|^{q_i-2} D^i T_M(u_n))_n$ is bounded in $(L^{q_i'}(Q_T))^N$, and

$$|D^i T_M(u_n)|^{q_i-2} D^i T_M(u_n) \to |D^i T_M(u)|^{q_i-2} D^i T_M(u) \text{ a.e. in } Q_T.$$

It follows that $|D^i T_M(u_n)|^{q_i-2} D^i T_M(u_n) \rightharpoonup |D^i T_M(u)|^{q_i-2} D^i T_M(u)$ in $L^{q_i'}(Q_T)$, and since $S''(u_n)\varphi D^i T_M(u_n) + S'(u_n) D^i\varphi \to S''(u)\varphi D^i T_M(u) + S'(u) D^i \varphi$ in $L^{q_i}(Q_T)$, we conclude that

$$\begin{aligned} \lim_{n\to+\infty} \int_{Q_T} & |D^i T_M(u_n)|^{q_i-2} D^i T_M(u_n) \cdot (S''(u_n)\varphi D^i T_M(u_n) + S'(u_n) D^i \varphi)\, dx\, dt \\ &= \int_{Q_T} |D^i u|^{q_i-2} D^i u \cdot (S''(u)\varphi D^i u + S'(u) D^i \varphi)\, dx\, dt. \end{aligned} \tag{37}$$

Moreover, since $S(u_n)\varphi \rightharpoonup S(u)\varphi$ weak-$\star$ in $L^\infty(Q_T)$, using (23) we get

$$\int_{Q_T} \gamma(x,t)|u_n|^{q_0-2} u_n S'(u_n)\varphi\, dx\, dt \to \int_{Q_T} \gamma(x,t)|u|^{q_0-2} u S'(u)\varphi\, dx\, dt. \tag{38}$$

With some modifications as in (33), we find

$$\int_{Q_T} \frac{f_n S'(u_n)\varphi}{\left(|T_n(u_n)| + \frac{1}{n}\right)^\theta}\, dx\, dt \to \int_{Q_T} \frac{f S'(u)\varphi}{u^\theta}\, dx\, dt. \tag{39}$$

By combining (35)-(39), we deduce that u is a renormalized solution to problem (2). ◄

References

[1] Abdelaziz, H., Mecheter, R. (2025) *Regularity results for a singular elliptic equation involving variable exponents*, Bol. Soc. Paran. Mat. (3s.), **43**, 1–25.

[2] Abdelaziz, H., Mokhtari, F. (2022) *Nonlinear anisotropic degenerate parabolic equations with variable exponents and irregular data*, Journal of Elliptic and Parabolic Equations, **8**, 513–532.

[3] Andreu, F., Mazón, J. M., Segura de León, S., Toledo, J. (1999) *Existence and uniqueness for a degenerate parabolic equation with L^1-data*, Transactions of the American Mathematical Society, **351(1)**, 285–306.

[4] Banks, H.T. (1975) *Modeling and control in the biomedical sciences*, Lecture Notes in Biomathematics, **6**, Springer, Berlin.

[5] Bendahmane, M., Chrif, M., El Manouni, S. (2011) *An approximation result in generalized anisotropic Sobolev spaces and applications*, Zeitschrift für Analysis und ihre Anwendungen, **30(3)**, 341–353.

[6] Benilan, P., Boccardo, L., Gallouët, T., Gariepy, R., Pierre, M., Vázquez, J.L. (1995) *An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations*, Annali della Scuola Normale Superiore di Pisa, Classe di Scienze, **4**, 241–273.

[7] Boccardo, L., Gallouët, T., Vázquez, J.L. (1991) *Some regularity results for nonlinear parabolic equations in L^1*, Rendiconti del Seminario Matematico dell'Università di Politecnico di Torino, Special Issue, 69–74.

[8] Chrif, M., El Manouni, S., Hjiaj, H. (2020) *Parabolic anisotropic problems with lower order terms and integrable data*, Differential Equations and Applications, **12(4)**, 411–442.

[9] De Bonis, I., De Cave, L.M. (2014) *Degenerate parabolic equations with singular lower order terms*, Differential and Integral Equations, **27**, 949–976.

[10] Di Castro, A. (2009) *Existence and regularity results for anisotropic elliptic problems*, Advanced Nonlinear Studies, **9(2)**, 367–393.

[11] DiPerna, R.J., Lions, P.-L. (1989) *On the Cauchy problem for Boltzmann equations: global existence and weak stability*, Annals of Mathematics, **130(2)**, 321–366.

[12] El Ouardy, M., El Hadfi, Y., Sbai, A. (2024) *Existence and regularity results of parabolic problems with convection term and singular nonlinearity*, Zeitschrift für Analysis und ihre Anwendungen, **43**, 299–328.

[13] Fragalà, I., Gazzola, F., Kawohl, B. (2004) *Existence and nonexistence results for anisotropic quasilinear elliptic equations*, Annales de l'Institut Henri Poincaré Analyse Non Linéaire, **21(5)**, 715–734.

[14] Fulks, W., Maybee, J.S. (1960) *A singular non-linear equation*, Osaka Journal of Mathematics, **12**, 1–19.

[15] Gatica, J.A., Oliker, V., Waltman, P. (1989) *Singular nonlinear boundary-value problems for second-order ordinary differential equations*, Journal of Differential Equations, **79**, 62–78.

[16] Ghergu, M., Radulescu, V. (2005) *Multi-parameter bifurcation and asymptotics for the singular Lane–Emden–Fowler equation with a convection term*, Proceedings of the Royal Society of Edinburgh Section A, **135(1)**, 61–83.

[17] Hewitt, E., Stromberg, K. (1965) *Real and abstract analysis*, Springer-Verlag, Berlin.

[18] Khelifi, H., Kokhtari, F. (2024) *Nonlinear degenerate parabolic equations with a singular nonlinearity*, Acta Applicandae Mathematicae, **189(1)**, Article 6.

[19] Mokhtari, F., Mecheter, R. (2019) *Anisotropic degenerate parabolic problems in* $\mathbb{R}^N$ *with variable exponent and locally integrable data*, Mediterranean Journal of Mathematics, **16(2)**, Article 61.

[20] Mokhtari, F. (2010) *Anisotropic parabolic problems with measure data*, Differential Equations and Applications, **2(1)**, 123–150.

[21] Mounim, E., El Hadfi, Y., Aziz, I. (2022) *Existence and regularity results for a singular parabolic equation with degenerate coercivity*, Discrete and Continuous Dynamical Systems Series S, **15**, 117–141.

[22] Nachman, A., Callegari, A. (1986) *A nonlinear singular boundary value problem in the theory of pseudoplastic fluids*, SIAM Journal on Applied Mathematics, **28**, 271–281.

[23] O'Regan, D. (1993) *Some general existence principles and results for* $(f(y'))' = qf(t, y, y')$, $0 < t < 1$, SIAM Journal on Mathematical Analysis, **24(3)**, 648–668.

[24] Panda, A., Choudhuri, D., Saoudi, K. (2024) *A parabolic problem involving* $p(x)$*-Laplacian, a power and a singular nonlinearity*, Journal of Elliptic and Parabolic Equations, **10**, 1153–1186.

[25] Simon, J. (1987) *Compact sets in the space* $L^p(0, T; B)$, Annali di Matematica Pura ed Applicata, **146**, 65–96.

[26] Zaater, W., Khelifi, H. (2024) *Bounded solutions in anisotropic degenerate parabolic problems with a singular term*, Indian Journal of Pure and Applied Mathematics. https://doi.org/10.1007/s13226-024-00691-4

[27] Zhang, C., Zhou, S. (2010) *Renormalized and entropy solutions for nonlinear parabolic equations with variable exponents and* L^1 *data*, Journal of Differential Equations, **248**, 1376–1400.

Hellal Abdelaziz
Laboratory of Functional Analysis and Geometry of Spaces, University of M'sila, University Pole, Road Bourdj Bou Arreiridj, M'sila 28000, Algeria
E-mail: abdelaziz.hellal@univ-msila.dz

Rabah Mecheter
Laboratory of Functional Analysis and Geometry of Spaces, University of M'sila, University Pole, Road Bourdj Bou Arreiridj, M'sila 28000, Algeria
E-mail: rabah.mecheter@univ-msila.dz

Received 03 September 2025
Accepted 07 November 2025