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Degenerated Boundary Conditions of a Sturm–Liouville Problem with a Potential–Distribution

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Abstract. We describe all degenerate boundary conditions in the homogeneous Sturm-Liouville problem with a Potential-Distribution. We show that for the case $y_1(x,\,\lambda) \equiv y_2^{[1]}(x,\,\lambda)$, the characteristic determinant is zero if and only if the boundary conditions are falsely periodic boundary conditions; the characteristic determinant is identically a nonzero constant if and only if the boundary conditions are generalized Cauchy conditions. For the case the case $y_1(x,\,\lambda) \not\equiv y_2^{[1]}(x,\,\lambda)$ in which the characteristic determinant is identically zero is impossible and that the only possible degenerate boundary conditions are the Cauchy conditions.

Key Words and Phrases: Sturm–Liouville problem, eigenvalues, degenerate boundary conditions, potential-distribution.

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1. Introduction

Degenerate boundary conditions are the boundary conditions such that the characteristic determinant of the corresponding eigenvalue problem is identically a constant [1, p. 35].

If the coefficients of an ordinary linear differential equation with two-point boundary conditions are continuous on the interval [0,1], then the following two cases are possible for the spectrum of the corresponding differential operator: (i) there exist at most countably many eigenvalues, which do not have limit points in $\mathbb{C}$; (ii) every $\lambda \in \mathbb{C}$ is an eigenvalue [2, pp. 13, 27].

Eigenvalue problems with nongenerated boundary conditions are sufficiently well studied (see [3, 4]).

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Degenerate boundary conditions less studied. It is well known, perhaps, only an example for differential operator D^2 for which the spectrum fills the entire complex plane [5]. In [6] (see also [7]) the example for differential operator of any even order for which the spectrum fills the entire complex plane is considered. This boundary conditions have the following form

$$U_j(y) = y^{(j-1)}(0) + (-1)^{j-1}\, y^{(j-1)}(1) = 0, \qquad j = 1, 2, ..., n. \tag{1}$$

It is shown in [8] that if $d \neq \pm 1$, $n = 2\nu$ with $\nu > 1$, then the spectrum of problem

$$y^{(n)}(x) + \sum_{m=1}^{n} p_m(x) y^{(n-m)}(x) + \lambda y(x) = 0, \quad y^{(2\nu-j)}(0) + d(-1)^{j+1} y^{(2\nu-j)}(\pi) = 0$$

is empty.

Until recently, the following question, posed in particular in [7], remained open: Do there exist spectral problems with an odd-order differential operator and with spectrum filling the entire complex plane? It was shown in [9] that there exist such operators.

Several problems related to degenerate boundary conditions were considered in [10, 11, 12].

The [13] found all the degenerate boundary conditions in the Sturm-Liouville problem for the case when the potential is a continuous function.

In this paper, all the degenerate boundary conditions in the Sturm-Liouville problem are found for the case when the potential is a distribution function.

Suppose that the potential q is a complex-valued generalized function such that $q = \sigma'$, where $\sigma \in L_2\,(0, 1)$, and the derivative σ' is understood in the sense of the theory of distributions. The Sturm-Liouville Problem with equation

$$-(y^{[1]})' - \sigma(x)\, y^{[1]} - \sigma^2(x)\, y, \quad y^{[1]} = y'(x) - \sigma(x)\, y(x). \tag{2}$$

and boundary conditions

$$U_i(y) = a_{i1}\, y^{[0]}(0) + a_{i2}\, y^{[1]}(0) + a_{i3}\, y^{[0]}(\pi) + a_{i4}\, y^{[1]}(\pi) = 0, \qquad i = 1, 2. \tag{3}$$

we call the problem L. For smooth functions $\sigma(x)$ we get the classical Sturm-Liouville problem.

There are a large number of papers (see, for example, [14, 15]) in which the equation is studied for $q(x) = \sum_j h_j\, \delta(x - x_j)$, where $h_j,\ x_j \in \mathbb{R}$, and δ is the Dirac function. Such potentials, with a suitable choice, can satisfy the conditions required in the article.

We denote the matrix consisting of the coefficients a_{lk} in the boundary conditions (3) by A and the minor consisting of the ith and jth columns of this matrix by M_{ij},

$$A = \left\| \begin{matrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{matrix} \right\|, \quad M_{ij} = \left| \begin{matrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{matrix} \right|, \qquad i,j = 1,2,3,4. \tag{4}$$

In what follows, we assume that the rank of the matrix A is equal to 2, rank$A$=2.

The eigenvalues of the problem L are the roots of the entire function ([1, p. 33–36]

$$\begin{aligned} \Delta(\lambda) = M_{12} + M_{34} + M_{32}\, y_1(\pi,\,\lambda) + M_{42}\, y_1^{[1]}(\pi,\,\lambda) + \\ + M_{13}\, y_2(\pi,\,\lambda) + M_{14}\, y_2^{[1]}(\pi,\,\lambda), \end{aligned} \tag{5}$$

where $y_1(x,\,\lambda)$ and $y_2(x,\,\lambda)$ are the linearly independent solutions of Eq. (2) satisfying the conditions

$$y_1^{[0]}(0,\,\lambda) = 1, \quad y_1^{[1]}(0,\,\lambda) = 0, \quad y_2^{[0]}(0,\,\lambda) = 0, \quad y_2^{[1]}(0,\,\lambda) = 1.$$

The asymptotic formulas

$$\begin{aligned} &y_1(x,\,\lambda) = \cos sx\,(1 + o(1)), \quad y_2(x,\,\lambda) = \tfrac{1}{s}\,\sin sx\,(1 + o(1)), \\ &y_1^{[1]}(x,\,\lambda) = -s\,\sin sx\,(1 + o(1)), \quad y_2^{[1]}(x,\,\lambda) = \cos sx\,(1 + o(1)), \end{aligned} \tag{6}$$

hold for sufficiently large s, $s = \sqrt{\lambda} \in \mathbb{R}$ [16].

2. The case $y_1(x,\,\lambda) \equiv y_2^{[1]}(x,\,\lambda)$

If $y_1(x,\,\lambda) \equiv y_2^{[1]}(x,\,\lambda)$ and $\Delta(\lambda) \equiv C = \text{const}$, then it follows from relations (5) and (6) that

$$M_{12} + M_{34} = C,\; M_{32} + M_{14} = 0,\; M_{42} = 0,\; M_{13} = 0. \tag{7}$$

In the classical case, when the quasi-derivative coincides with the ordinary derivative, the identity is $y_1(\pi,\,\lambda) \equiv y_2'(\pi,\,\lambda)$ is true if and only if $q(x) = q(\pi - x)$ almost everywhere on $[0,\pi]$ [17, Lemma 4].

To find the minors M_{12} and M_{34}, we use the fact that the minors of a matrix cannot be arbitrary numbers. Given numbers M_{12}, M_{13}, M_{14}, M_{23}, M_{24}, and M_{34} are the minors of some matrix if and only if the following Plücker relations hold [18]:

$$M_{12}\, M_{34} - M_{13}\, M_{24} + M_{14}\, M_{23} = 0. \tag{8}$$

[The minors M_{23} and M_{24} occurring in relations (8) differ from the minors M_{32} and M_{42} in relations (7) only in sign.] From relations (7) and (8), we obtain two sets of minors,

$$M_{12} = C_1, \quad M_{34} = C - C_1, \quad M_{32} = \mp\sqrt{C_1\,(C_1 - C)}, \\ M_{42} = 0, \quad M_{13} = 0, \quad M_{14} = \pm\sqrt{C_1\,(C_1 - C)}. \tag{9}$$

If C = 0 [the case in which $\Delta(\lambda) \equiv 0$], then from (9) we obtain the relations

$$M_{12} = C_1, \quad M_{34} = -C_1, \quad M_{32} = \mp C_1, \quad M_{42} = 0, \quad M_{13} = 0, \quad M_{14} = \pm C_1. \tag{10}$$

By using methods for the reconstruction of a matrix from its minors [18], for these sets of minors we uniquely find two forms of boundary conditions

$$y(0) \mp y(\pi) = 0, \quad y^{[1]}(0) \pm y^{[1]}(\pi) = 0. \tag{11}$$

Conditions (11) are said to be falsely periodic, because they are degenerate and differ from non- degenerate periodic or antiperiodic boundary conditions by the change of only one sign (plus is replaced by minus, or minus is replaced by plus).

If $C \neq 0$ [the case in which $\Delta(\lambda) \not\equiv 0$], then the form of boundary conditions depends on nonzero minors in (9). If $C - C_1 = 0$, then we obtain the Cauchy conditions $y(0) = y^{[1]}(0) = 0$; if $C_1 = 0$, then we have the Cauchy conditions $y(\pi) = y^{[1]}(\pi) = 0$; and if $C - C_1 \neq 0$ and $C_1 \neq 0$, then we obtain the conditions

$$y(0) \mp a\,y(\pi) = 0, \quad y^{[1]}(0) \pm a\,y^{[1]}(\pi) = 0, \tag{12}$$

where $a = \sqrt{\frac{C_1 - C}{C_1}}$. We have thereby proved the following assertion.

Theorem 1. *If $y_1(x,\,\lambda) \equiv y_2^{[1]}(x,\,\lambda)$, then the condition $\Delta(\lambda) \equiv 0$ is realized only in the case where the boundary conditions (3) are the falsely periodic boundary conditions (11), and the case $\Delta(\lambda) \equiv C \neq 0$ is realized only in the case where conditions (3) are generalized Cauchy conditions, i.e., conditions of the form (12), where $0 \leq a \leq \infty$ and $a \neq 1$.*

3. The case $y_1(x,\,\lambda) \not\equiv y_2^{[1]}(x,\,\lambda)$

If $y_1(x,\,\lambda) \not\equiv y_2^{[1]}(x,\,\lambda)$ and $\Delta(\lambda) \equiv C = \text{const}$, then it follows from relations (5) and (6) that

$$1) \quad M_{12} = C \neq 0, \quad M_{34} = 0, \quad M_{32} = 0, \quad M_{42} = 0, \quad M_{13} = 0, \quad M_{14} = 0; \tag{13}$$

2) $M_{12}=0, \quad M_{34}=C\neq 0, \quad M_{32}=0, \quad M_{42}=0, \quad M_{13}=0, \quad M_{14}=0.$ (14)

The case in which $C=0$ (and hence $\Delta(\lambda)\equiv 0$) cannot be realized, because the vanishing of all second-order determinants contradicts the condition rank$A$=2. With the use of methods for the reconstruction of a matrix from its minors [18] , the sets of minors (13) and (14) uniquely determine the boundary conditions (3) (i.e., the matrix A can be found up to a linear transformation of its rows). The set of minors (13) corresponds to the Cauchy conditions $y(0)=y^{[1]}(0)=0$, and the set of minors (14) corresponds to the Cauchy conditions $y(\pi)=y^{[1]}(\pi)=0$. We have thereby proved the following assertion.

Theorem 2. *If $y_1(x,\lambda)\not\equiv y_2^{[1]}(x,\lambda)$, then the case in which $\Delta(\lambda)\equiv 0$ is impossible, and the Cauchy conditions $y(0)=y^{[1]}(0)=0$ and $y(\pi)=y^{[1]}(\pi)=0$ are the only possible degenerate boundary conditions.*

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