Azerbaijan Journal of Mathematics
V. 9, No 1, 2019, January
ISSN 2218-6816

Direct and Boundary Inverse Spectral Problems for Sturm-Liouville Differential Operators on Noncompact Star-Shaped Graphs

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Abstract. This work is dedicated to the solution of direct and inverse spectral problems for Sturm-Liouville differential operators on star-shaped graphs with $n+1$ edges; n edges are finite, and one edge is infinite. Eigenvalues visible at infinity, eigenvalues invisible at infinity and resonances of this spectral problem are found. It is shown that the problem of determining n unknown coefficients of boundary conditions by n eigenvalues visible at infinity and (or) resonances has $n!$ solutions. If the finite edges of the graph are equal in length, then the boundary conditions are determined up to a permutation of the fastening ends. It is proved that the boundary conditions are uniquely determined up to permutations on the dead-end vertices by resonances. If $2n+1$ eigenvalues visible at infinity and (or) resonances are used to reconstruct the boundary conditions ($2n$ unknown coefficients), then the identification problem of the boundary conditions has $n!$ solutions. If the finite edges of the graph are equal in length, then the boundary conditions are uniquely determined up to permutations on the dead-end vertices.

Key Words and Phrases: eigenvalues, graph, inverse problems, Kirchhoff boundary conditions, Jost solution.

2010 Mathematics Subject Classifications: 34A55, 58C40

1. Introduction

Solutions to inverse spectral problems for Sturm-Liouville operators on an interval are presented in the monographs [10, 11] and other works. Differential operators on graphs (networks, trees) often appear in natural sciences and engineering (see [8, 5, 6, 7, 9], [17, 18, 19, 20] and the references therein). Most of the results in this direction are dedicated to direct problems of studying properties of the spectrum and the root functions for operators on graphs. Inverse spectral problems, because of their nonlinearity, are more difficult to investigate, and nowadays there exists only a small number of papers in this area. In particular, some inverse spectral problems of recovering the coefficients of differential operators on trees (i.e on graphs without cycles) were solved in [2, 3, 4], [14, 15]. Inverse problems for Sturm-Liouville operators on graphs with a cycle were studied in [1, 12, 13, 16, 21, 22, 23].

2. General solution

Let $\Gamma = \gamma_1 \times ... \times \gamma_n \times \gamma_{n+1}$ be a star-shaped graph which consists of n finite rays $\gamma_j = \{x_j \in (0, l_j)\}, j = 1, ..., n$ and one semi-infinite ray $\gamma_{n+1} = \mathbf{R}_+ = \{x_{n+1} \in (0, \infty)\}$, with the origin of each ray identified with the single vertex of the graph. We consider on Γ the equation

$$Ly_j = -\frac{d^2 y_j}{dx_j^2} + q(x_j) y_j = \mu^2 y_j(x), \quad x_j \in \gamma_j, \tag{1}$$

defined for functions y satisfying the natural Kirchhoff boundary conditions on the vertex:

$$y_1(0) = ... = y_n(0) = y_{n+1}(0), \tag{2}$$

$$y_1'(0) + ... + y_n'(0) + y_{n+1}'(0) = 0, \tag{3}$$

We further require the boundary conditions on the end-points $x_j = l_j$:

$$y_j'(l_j) + H_j y_j(l_j) = 0, \ j = 1, \cdots, n. \tag{4}$$

Here

$$H_j = h_j - \mu^2 m_j \tag{5}$$

and μ is a spectral parameter $\Im\mu \geq 0$.

Each potential $q_j(x) \in L(\gamma_j)$ $(j = 1, \cdots, n+1)$.

Let $s_j(x, \mu),\ c_j(x, \mu)$, $j = 1, ..., n+1$, be the solution of (1.1) on the γ_j which satisfies the initial condition $s_j(0, \mu) = 1 - c_j(0, \mu) = 0$, $1 - s_j'(0, \mu) = c_j'(0, \mu) = 0$ $(j = 1, \cdots, n+1)$.

Let $e(x, \mu)$, be the Jost solution of (1.1) which satisfies the asymptotic condition $e(x, \lambda) \to e^{i\mu x}$ $(x \in \gamma_{n+1},\ x \to \infty)$.

General solution on every interval can be represented as

$$y_j(x, \mu) = \alpha_j(\mu) c_j(x, \mu) + \beta_j(\mu) s_j(x, \mu), \ j = 1, \cdots, n, \tag{6}$$

$$y_{n+1}(x, \mu) = \Delta(\mu) s_{n+1}(x, \mu) + \rho(\mu) e(x, \mu) \text{ in the case } e(0, \mu) \neq 0 \tag{7}$$

$$y_{n+1}(x, \mu) = \Delta(\mu) c_{n+1}(x, \mu) + \rho(\mu) e(x, \mu) \text{ in the case } e(0, \mu) = 0. \tag{8}$$

We look for a general solution of (1)–(4) for $\Im s \geq 0$.

It has to satisfy the first Kirchhoff condition (2) for the first n components (6):

$$\alpha_1(\mu) = \cdots = \alpha_n(\mu) \tag{9}$$

We denote $\alpha(\mu) = \alpha_1(\mu) = \cdots = \alpha_n(\mu)$.

Boundary conditions on the ends of finite intervals (4) imply the following conditions on $\alpha(\mu), \beta_1(\mu), \cdots, \beta_2(\mu)$:

$$\alpha(\mu)\left[c_j'(l_j,\mu) + H_j c_j(l_j,\mu)\right] + \beta_j(\mu)\left[s_j'(l_j,\mu) + H_j s_j(l_j,\mu)\right] = 0,\ j = 1,\cdots,n. \tag{10}$$

We define $(i = 1, \cdots, n)$

$$\sigma_j(\mu) = s_j'(l_j,\mu) + H_j s_j(l_j,\mu). \tag{11}$$

$$\kappa_j(\mu) = c_j'(l_j,\mu) + H_j c_j(l_j,\mu). \tag{12}$$

Then the condition (11) can be rewritten as

$$\alpha(\mu)\kappa_j(\mu) + \beta_j(\mu)\sigma_j(\mu) = 0,\ j = 1,\cdots,n. \tag{13}$$

It is important to mention, that $\sigma_j(\mu)$ and $\kappa_j(\mu)$ cannot vanish simultaneously: multiply (11) by $c_j(l_j,\mu)$ and subtract (12) multiplied by $s_j(l_j,\mu)$. As a result:

$$\sigma_j(\mu)c_j(l_j,\mu) - \kappa_j(\mu)s_j(l_j,\mu) = s_j'(l_j,\mu)c_j(l_j,\mu) - c_j'(l_j,\mu)s_j(l_j,\mu) = 1 \tag{14}$$

A. We first consider the case

$$e(0,\mu) \neq 0. \tag{15}$$

In this case, the general solution on the infinite part γ_{n+1} is given by (7)

$$y_{n+1}(x,\mu) = \Delta(\mu)s_{n+1}(x,\mu) + \rho(\mu)e(x,\mu)$$

and it follows from (2), (7), (9) that

$$\alpha(\mu) = \rho(\mu)e(0,\mu) \tag{16}$$

and

$$\rho(\mu) = \frac{\alpha(\mu)}{e(0,\mu)}. \tag{17}$$

The second Kirchhoff condition (3) requires (in view of (7))

$$\beta_1(\mu) + \cdots + \beta_n(\mu) + \Delta(\mu) + \rho(\mu)e'(0,\mu) = 0, \tag{18}$$

i.e.

$$\beta_1(\mu) + \cdots + \beta_n(\mu) + \Delta(\mu) + \alpha(\mu)\frac{e'(0,\mu)}{e(0,\mu)} = 0. \tag{19}$$

A1. Consider the case

$$\sigma_1(\mu) \times \cdots \times \sigma_n(\mu) \neq 0. \tag{20}$$

Then it follows from (10) that

$$\beta_j(\mu) = -\alpha(\mu)\frac{c_j'(l_j,\mu) + H_j c_j(l_j,\mu)}{s_j'(l_j,\mu) + H_j s_j(l_j,\mu)} = -\alpha(\mu)\frac{\kappa_j(\mu)}{\sigma_j(\mu)}. \tag{21}$$

It follows from (19) that

$$\Delta(\mu) = \alpha(\mu)\left[\sum_{j=1}^{n}\frac{\kappa_j(\mu)}{\sigma_j(\mu)} - \frac{e'(0,\mu)}{e(0,\mu)}\right]. \tag{22}$$

So the general solution to the problem (1)-(4) in the case of spectral parameter s such that $\Im s \geq 0$, and $e(0,\mu)\sigma_1(\mu) \times \cdots \times \sigma_n \neq 0$ is given by

$$y_j(x,\mu) = \alpha(\mu)\left(c_j(x,\mu) - \frac{\kappa_j(\mu)}{\sigma_j(\mu)}s_j(x,\mu)\right), \; j = 1, \cdots, n, \tag{23}$$

$$y_{n+1}(x,\mu) = \alpha(\mu)\left(\left[\sum_{j=1}^{n}\frac{\kappa_j(\mu)}{\sigma_j(\mu)} - \frac{e'(0,\mu)}{e(0,\mu)}\right]s_{n+1}(x,\mu) + \frac{e(x,\mu)}{e(0,\mu)}\right), \tag{24}$$

where $\alpha(\mu)$ is arbitrary.

A2. Consider the case

$$\sigma_1(\mu) \times \cdots \times \sigma_n(\mu) = 0. \tag{25}$$

Denote

$$J = \{j : \sigma_j(\mu) = 0.\} \tag{26}$$

Then $\kappa_j(\mu) \neq 0$, for$j \in J$, and, to fulfil the (10) and, according to (17), it has to be

$$\alpha(\mu) = 0 = \rho(\mu) \tag{27}$$

So the general solution in this case has to be in the form

$$y_j(x,\mu) = \beta_j(\mu)s_j(x,\mu), \; j = 1, \cdots, n, \tag{28}$$

$$y_{n+1}(x,\mu) = \Delta(\mu) s_{n+1}(x,\mu). \tag{29}$$

But $\sigma_j(\mu) \neq 0, j \notin J$, and as a result, to satisfy (10), it has to be $\beta_j = 0$. It follows from (19) that $\Delta(\mu) = -\sum_{j\in J} \beta_j(\mu)$ and, as a result, the general solution in this case is

$$y_j(x,\mu) = \beta_j(\mu) s_j(x,\mu),\ j \in J, \tag{30}$$

$$y_j(x,\mu) \equiv 0, j \notin J, \tag{31}$$

$$y_{n+1}(x,\mu) = -\left(\sum_{j\in J} \beta_j(\mu)\right) s_{n+1}(x,\mu), \tag{32}$$

where $\beta_j(\mu)$ are arbitrary.

B. Now we investigate the case

$$e(0,\mu) = 0. \tag{33}$$

In this case the general solution on the infinite part γ_{n+1} is given by (8)

$$y_{n+1}(x,\mu) = \Delta(\mu) c_{n+1}(x,\mu) + \rho(\mu) e(x,\mu)$$

and it follows from (2), (9), (8) (33) that

$$\alpha(\mu) = \Delta(\mu). \tag{34}$$

The second Kirchhoff condition (3) requires (in view of (8))

$$\beta_1(\mu) + \cdots + \beta_n(\mu) + \rho(\mu) e'(0,\mu) = 0, \tag{35}$$

B1. Consider the case

$$\sigma_1(\mu) \times \cdots \times \sigma_n(\mu) \neq 0. \tag{36}$$

Then it follows from (10) that

$$\beta_j(\mu) = -\alpha(\mu) \frac{c_j'(l_j,\mu) + H_j c_j(l_j,\mu)}{s_j'(l_j,\mu) + H_j s_j(l_j,\mu)} \tag{37}$$

From (35) we have

$$\rho(\mu) = \frac{\alpha(\mu)}{e'(0,\mu)} \sum_{j=1}^{n} \frac{c_j'(l_j,\mu) + H_j c_j(l_j,\mu)}{s_j'(l_j,\mu) + H_j s_j(l_j,\mu)}. \tag{38}$$

So the general solution to the problem (1)-(4) in the case of spectral parameter s such that $\Im s \geq 0$, and $e(0,\mu)=0,\ \sigma_1(\mu)\times\cdots\times\sigma_n(\mu)\neq 0$ is given by

$$y_j(x,\mu)=\alpha(\mu)\left(c_j(x,\mu)-\frac{\kappa_j(\mu)}{\sigma_j(\mu)}s_j(x,\mu)\right),\ j=1,\cdots,n, \tag{39}$$

$$y_{n+1}(x,\mu)=\alpha(\mu)\left(c_3(x,\mu)+\left[\sum_{j=1}^{n}\frac{\kappa_j(\mu)}{\sigma_j(\mu)}\right]\frac{e(x,\mu)}{e'(0,\mu)}\right), \tag{40}$$

where $\alpha(\mu)$ is arbitrary.

B2. Consider the case

$$\sigma_1(\mu)\times\cdots\times\sigma_n(\mu)=0. \tag{41}$$

Denote

$$J=\{j:\sigma_j(\mu)=0\}. \tag{42}$$

Then $\kappa_j(\mu)\neq 0$, for $j\in J$, and, to fulfil the (10) and, according to (17), it has to be

$$\alpha(\mu)=0=\Delta(\mu) \tag{43}$$

So the general solution in this case has to be in the form

$$y_j(x,\mu)=\beta_j(\mu)s_j(x,\mu),\ j=1,\cdots,n. \tag{44}$$

$$y_{n+1}(x,\mu)=\rho(\mu)e(x,\mu). \tag{45}$$

But $\sigma_j(\mu)\neq 0, j\notin J$, and as a result, to satisfy (10), it has to be $\beta_j=0$. It follows from (35) that $\rho(\mu)=-\left(\sum_{j\in J}\beta_j(\mu)\right)\frac{e(x,\mu)}{e'(0,\mu)}$ and, as a result, the general solution in such case is

$$y_j(x,\mu)=\beta_j(\mu)s_j(x,\mu),\ j\in J \tag{46}$$

$$y_j(x,\mu)\equiv 0, j\notin J \tag{47}$$

$$y_{n+1}(x,\mu)=-\left(\sum_{j\in J}\beta_j(\mu)\right)\frac{e(x,\mu)}{e'(0,\mu)}, \tag{48}$$

where $\beta_j(\mu)$ are arbitrary.

3. Eigenvalues visible at infinity

Eigenvalues are called "visible at infinity" if there exist corresponding eigenfunctions which are not identically vanishing on γ_{n+1}.

It follows from representations (7) and (8) that all eigenvalues λ visible at infinity ($\lambda = \mu^2$, $\Im\mu > 0$) have to satisfy $\Delta(\mu) = 0$ and $\rho(\mu) \neq 0$. We see that these are the cases A1 with $\Delta(\mu) = 0$ and the case B2.

Theorem 1. *Eigenvalues $\lambda = \mu^2$, $\Im\mu > 0$ "visible at infinity" constitute the set $\lambda = \mu^2$, $\mu \in V = V_1 \cup V_2$ with*

$$V_1 = \{\Im\mu > 0 : \; e(0,\mu)\sigma_1(\mu) \times \cdots \times \sigma_n(\mu) \neq 0, \; \Delta(\mu) = 0\}, \tag{49}$$

$$V_2 = \{\Im\mu > 0 : \; |e(0,\mu)| + |\sigma_1(\mu) \times \cdots \times \sigma_n(\mu)| = 0, \; \}. \tag{50}$$

Constants $\sigma_j(\mu)$ are defined by (11):

$$\sigma_j(\mu) = s'_j(l_j,\mu) + H_j s_j(l_j,\mu)$$

and $\Delta(\mu)$ is given by (1) with $\alpha(\mu) = 1$:

$$\Delta(\mu) = \alpha(\mu)\left[\sum_{j=1}^{n} \frac{\kappa_j(\mu)}{\sigma_j(\mu)} - \frac{e'(0,\mu)}{e(0,\mu)}\right],$$

where $\kappa_j(\mu)$ is given by (12):

$$\kappa_j(\mu) = c'_j(l_j,\mu) + H_j c_j(l_j,\mu).$$

4. Eigenvalues invisible at infinity

Eigenfunctions, corresponding to eigenvalues "invisible at infinity", have a part $y_{n+1}(x,\mu) \equiv 0$ on the infinite part γ_{n+1} (and $|y_1(x,\mu)| + \cdots + |y_n(x,\mu)| \not\equiv 0$). It means that this is a case where there exist at least two different i and j such that $\sigma_i(\mu) = \sigma_j(\mu) = 0$

Let us mention that the eigenvalue can be "visible" and "invisible" at the same time (the case B2 with at least two different i and j such that $\sigma_i(\mu) = \sigma_j(\mu) = 0$)

Theorem 2. *Eigenvalues $\lambda = \mu^2$, $\Im\mu \geq 0$ "invisible at infinity" constitute the set*

$$Iv = \{\Im s \geq 0 : \; |\sigma_i(\mu)| + |\sigma_j(\mu)| = 0, i \neq j\}. \tag{51}$$

5. Resonances

Let $q(x)$ be compact-supported. Then Jost solution $e(x,\mu)$ is defined on the whole μ-plane. We denote by "resonances" such values of μ on lower half-plane $\Im\mu < 0$ that there exists the solution to (1)–(4), which coincides with $e(x,\mu)$ on γ_3. Then we have

Theorem 3. *Resonances* $\lambda = \mu^2,\ \Im\mu < 0$ *constitute the set* $\lambda = \mu^2,\ \mu \in R = R_1 \cup R_2$ *with*

$$R_1 = \{\Im s < 0:\ e(0,\mu)\sigma_1(\mu) \times \cdots \times \sigma_n(\mu) \neq 0,\ \Delta(\mu) = 0\}\,, \tag{52}$$

$$R_2 = \{\Im s < 0:\ |e(0,\mu)| + |\sigma_1(\mu) \times \cdots \times \sigma_n(\mu)| = 0,\ \}\,. \tag{53}$$

6. Identification of boundary conditions coefficients for infinite star graph by eigenvalues and resonances

Let us consider the eigenvalues invisible at infinity or resonances μ_k such that $e(0,\mu_k) \neq 0$.

The coefficients of boundary conditions for finite edges can be found by the eigenvalues visible at infinity and resonances. From Section 3 (Eigenvalues Visible at Infinity) it follows that the eigenvalues visible at infinity ($\lambda = \mu^2,\ \Im\mu > 0$) have to satisfy

$$\sum_{j=1}^{n} \frac{c_j'(l_j,\mu) + H_j c_j(l_j,\mu)}{s_j'(l_j,\mu) + H_j s_j(l_j,\mu)} - \frac{e'(0,\mu)}{e(0,\mu)} = 0, \tag{54}$$

where

$$H_j = h_j - \mu^2 m_j.$$

The first boundary inverse problem. Given n eigenvalues of the problems visible at infinity and (or) resonances, n coefficients m_j, find n coefficients h_j.

The second boundary inverse problem. Given n eigenvalues of the problems visible at infinity and (or) resonances, n coefficients h_j, find n coefficients μ_j.

The third boundary inverse problem. Given $2n+1$ eigenvalues of the problems visible at infinity and (or) resonances, find n coefficients h_j and n coefficients m_j.

If $l_j = l$, $q_j(x) = q(x)$ and $m_j = m$ for the first problem ($h_j = h$ for the second problem), then it follows that $s_j(x,\mu) = s(x,\mu)$, $c_j(x,\mu) = c(x,\mu)$.

Putting the sum in (54) into a common denominator, we obtain for the first problem

$$\sum_{j=1}^{n}\big(c'(l,\mu)+(h_j-\mu^2\,m)\,c(l,\mu)\big)\prod_{\substack{i=1\\ i\neq j}}^{n}\big(s'(l,\mu)+(h_i-\mu^2\,m)\big)-$$
$$-\tfrac{e'(0,\mu)}{e(0,\mu)}\prod_{i=1}^{n}\big(s'(l,\mu)+(h_i-\mu^2\,m)\,s(l,\mu)\big)=0,$$

From this it follows that

$$\sum_{j=1}^{n}\Big(c'(l,\mu)+(h_j-\mu^2\,m)\,c(l,\mu)-\tfrac{e'(0,\mu)}{e(0,\mu)}\,(s'(l,\mu)+(h_j-\mu^2\,m)\,s(l,\mu)\Big)\times$$
$$\times\tfrac{1}{n}\prod_{\substack{i=1\\ i\neq j}}^{n}\big(s'(l,\mu)+(h_i-\mu^2\,m)\,s(l,\mu)\big)=0,$$

or

$$\Big(c(l,\mu)-\tfrac{e'(0,\mu)}{e(0,\mu)}\,s(l,\mu)\Big)\;s^{n-1}(l,\mu)\prod_{j=1}^{n}h_j+$$
$$+\sum_{j=1}^{n}\Big(c'(l,\mu)-\mu^2\,m\,c(l,\mu)-\tfrac{e'(0,\mu)}{e(0,\mu)}\,(s'(l,\mu)-\mu^2\,m\,s(l,\mu)\Big)\times$$
$$\times s^{n-1}(l,\mu)\cdot\prod_{\substack{i=1\\ i\neq j}}^{n}h_j+$$
$$+\sum_{j=1}^{n}\Big(c(l,\mu)-\tfrac{e'(0,\mu)}{e(0,\mu)}\;s(l,\mu)\Big)h_j\times$$
$$\times\sum_{\substack{i=1\\ i\neq j}}^{n}s^{n-2}(l,\mu)\cdot\big(s'(l,\mu)-\mu^2\,m\,s(l,\mu)\big)\cdot\prod_{\substack{i=1\\ i\neq j\\ i\neq k}}^{n}h_i$$
$$+\ldots+$$
$$+\sum_{j=1}^{n}\Big(c'(l,\mu)-\mu^2\,m\,c(l,\mu)-\tfrac{e'(0,\mu)}{e(0,\mu)}\,(s'(l,\mu)-\mu^2\,m\,s(l,\mu)\Big)\times$$
$$\times\tfrac{1}{n}\prod_{\substack{i=1\\ i\neq j}}^{n}\big(s'(l,\mu)-\mu^2\,m\,s(l,\mu)\big)=0.$$

So

$$\Big(c(l,\mu) - \tfrac{e'(0,\mu)}{e(0,\mu)}\, s(l,\mu)\Big)\ s^{n-1}(l,\mu)\ \prod_{j=1}^{n} h_j +$$
$$+\sum_{j=1}^{n}\Big(c'(l,\mu) - \mu^2\, m\, c(l,\mu) - \tfrac{e'(0,\mu)}{e(0,\mu)}\,(s'(l,\mu) - \mu^2\, m\, s(l,\mu)\Big)\times$$
$$\times s^{n-1}(l,\mu)\cdot \prod_{\substack{i=1\\ i\neq j}}^{n} h_j +$$
$$+\sum_{i=1}^{n}\Big(c(l,\mu) - \tfrac{e'(0,\mu)}{e(0,\mu)}\ s(l,\mu)\Big)\times$$
$$\times n\, s^{n-2}(l,\mu)\cdot\big(s'(l,\mu) - \mu^2\, m\, s(l,\mu)\big)\cdot \prod_{\substack{i=1\\ i\neq k}}^{n} h_i$$
$$+\ldots+$$
$$+\sum_{j=1}^{n}\Big(c'(l,\mu) - \mu^2\, m\, c(l,\mu) - \tfrac{e'(0,\mu)}{e(0,\mu)}\,(s'(l,\mu) - \mu^2\, m\, s(l,\mu)\Big)\times$$
$$\times\tfrac{1}{n}\prod_{\substack{i=1\\ i\neq j}}^{n}\big(s'(l,\mu) - \mu^2\, m\, s(l,\mu)\big) = 0,$$

or

$$\Big(c(l,\mu) - \tfrac{e'(0,\mu)}{e(0,\mu)}\, s(l,\mu)\Big)\ s^{n-1}(l,\mu)\ \prod_{j=1}^{n} h_j +$$
$$+\Big(c'(l,\mu) - \mu^2\, m\, c(l,\mu) - \tfrac{e'(0,\mu)}{e(0,\mu)}\,(s'(l,\mu) - \mu^2\, m\, s(l,\mu) +$$
$$+\big(c(l,\mu) - \tfrac{e'(0,\mu)}{e(0,\mu)}\ s(l,\mu)\big)\times$$
$$\times n\, s^{n-2}(l,\mu)\cdot\big(s'(l,\mu) - \mu^2\, m\, s(l,\mu)\big)s^{n-1}(l,\mu)\Big)\cdot\sum_{j=1}^{n}\prod_{\substack{i=1\\ i\neq j}}^{n} h_j + \ldots +$$
$$+\sum_{j=1}^{n}\Big(c'(l,\mu) - \mu^2\, m\, c(l,\mu) - \tfrac{e'(0,\mu)}{e(0,\mu)}\,(s'(l,\mu) - \mu^2\, m\, s(l,\mu)\Big)\times$$
$$\times\tfrac{1}{n}\prod_{\substack{i=1\\ i\neq j}}^{n}\big(s'(l,\mu) - \mu^2\, m\, s(l,\mu)\big) = 0,$$

If we know n eigenvalues visible at infinity and (or) resonances μ_k of the problem, then from this we obtain the system of n linear equations

$$\begin{gathered}\left(c(l,\mu_k)-\tfrac{e'(0,\mu_k)}{e(0,\mu)}\,s(l,\mu_k)\right)\ s^{n-1}(l,\mu_k)\ \prod_{j=1}^{n}h_j+\\ +\Bigg(c'(l,\mu)-\mu^2\,m\,c(l,\mu)-\tfrac{e'(0,\mu)}{e(0,\mu)}\,(s'(l,\mu)-\mu^2\,m\,s(l,\mu)+\\ +\big(c(l,\mu)-\tfrac{e'(0,\mu)}{e(0,\mu)}\ s(l,\mu)\big)\times\\ \times n\,s^{n-2}(l,\mu)\cdot\big(s'(l,\mu)-\mu^2\,m\,s(l,\mu)\big)s^{n-1}(l,\mu)\Bigg)\cdot\sum_{j=1}^{n}\prod_{\substack{i=1\\ i\neq j}}^{n}h_j+\\ +\ldots+\\ +\sum_{j=1}^{n}\left(c'(l,\mu_k)-\mu_k^2\,m\,c(l,\mu_k)-\tfrac{e'(0,\mu_k)}{e(0,\mu_k)}\,(s'(l,\mu_k)-\mu^2\,m\,s(l,\mu_k)\right)\times\\ \times\tfrac{1}{n}\prod_{\substack{i=1\\ i\neq j}}^{n}\big(s'(l,\mu_k)-\mu_k^2\,m\,s(l,\mu_k)\big)=0,\qquad k=1,2,\ldots,n.\end{gathered}\tag{55}$$

with respect to the unknowns

$$x_1=\prod_{j=1}^{n}h_j,\quad x_2=\sum_{j=1}^{n}\prod_{\substack{i=1\\ i\neq j}}^{n}h_i,\quad \ldots,\quad x_n=\sum_{j=1}^{n}h_j.\tag{56}$$

If the determinant of the system of linear equations (55) with respect to the unknowns x_i $(i=1,2,\ldots,n)$is not equal to zero, then the system of linear equations (55) has a unique solution with respect to the unknowns x_i $(i=1,2,\ldots,n)$. This unique solution can be found by Cramer's formulas.

The n coefficients h_j of the boundary conditions are determined ambiguously. They can be found from the system of nonlinear equations (56) from x_i by the Vieta's formulas.

It is easy to see that if you rearrange any numbers h_j, then the equations (56) remain the same. Therefore, the system of equations (56) has $n!$ solutions. These solutions h_j are determined up to permutations by their places.

Thus we have the following theorem.

Theorem 4. *If $l_j=l$, $m_j=m$, $q_j(x)=q(x)$, $j=1,2,\ldots,n$, $q_{n+1}(x)$ is compact supported, $e(0,\mu_k)\neq 0$, and the determinant of the system of linear equations (55) with respect to the unknowns x_i $(i=1,2,\ldots,n)$ is not equal to zero, then the first problem of identifying the coefficients h_j of the boundary conditions (4)* *has $n!$ solutions, which are determined up to permutations of the coefficients h_j in places.*

Similarly we obtain the following theorem.

Theorem 5. *If $l_j = l$, $h_j = h$, $q_j(x) = q(x)$, $j = 1, 2, \ldots, n$, $q_{n+1}(x)$ is compact supported, $e(0, \mu_k) \neq 0$, and the determinant of the system of linear equations (55) with respect to the unknowns x_i $(i = 1, 2, \ldots, n)$ is not equal to zero, then the second problem of identifying the coefficients m_j of the boundary conditions (4) has $n!$ solutions, which are determined up to permutations of the coefficients m_j in places.*

If, however, m_j (h_j) are different, then the identification problem h_j (m_j) also has $n!$ solutions. However, these coefficients are no longer located within a permutation of their positions. Let us show the solution of these problems by the examples of a star-shaped graph which consists of two finite ray $\gamma_j = \{x_j \in (0, 1)\}, j = 1, 2$ and one semi-infinite rays $\gamma_3 = R_+ = \{x_3 \in (0, \infty)\}$.

Let $\Gamma = \gamma_1 \times \gamma_2 \times \gamma_3$ be a star-shaped graph which consists of two finite ray $\gamma_j = \{x_j \in (0, 1)\}, j = 1, 2$ and one semi-infinite rays $\gamma_3 = R_+ = \{x_3 \in (0, \infty)\}$, with the origin of each ray identified with the single vertex of the graph. We consider on Γ the equation

$$Ly_j = -\frac{d^2 y_j}{dx_j^2} + q(x_j) y_j = \mu^2 y_j(x), \quad x_j \in \gamma_j \tag{57}$$

for functions y_j satisfying the natural Kirchhoff boundary conditions on the vertex

$$y_1(0) = y_2(0) = y_3(0), \quad y_1'(0) + y_2'(0) + y_3'(0) = 0, \tag{58}$$

and the boundary conditions

$$y_j'(1) + H_j\, y_j(1) = 0, \quad H_j = h_j - m_j\, \mu^2, \quad j = 1, 2. \tag{59}$$

Let $s_j(x, \mu)$, $c_j(x, \mu)$, $j = 1, 2, 3$, be the solution of (57) which satisfies the initial condition $s_j(0, \mu) = 1 - c_j(0, \mu) = 0$, $1 - s_j^{'}(0, \mu) = c_j^{'}(0, \mu) = 0$ $(j = 1, 2, 3)$.

Let $e(x, \mu)$ be the Jost solution of (57) which satisfies the asymptotic condition $e(x, \lambda) \to e^{i\,\mu\, x}$ $(x \in \gamma_3,\ x \to \infty)$.

It follows from (54) that all eigenvalues visible at in infinity have to be such that $\Im s > 0$, and in the case $e(0, \mu) \neq 0$

$$\frac{c_1'(1, \mu) + H_1\, c_1(1, \mu)}{s_1'(1, \mu) + H_1\, s_1(1, \mu)} + \frac{c_2'(1, \mu) + H_2\, c_2(1, \mu)}{s_2'(1, \mu) + H_2\, s_2(1, \mu)} - \frac{e'(0, \mu)}{e(0, \mu)} = 0.$$

If we know two eigenvalues of problem (57)–(59) we can find two coefficients of boundary conditions (59). Let s_1^2 and s_2^2 be the eigenvalues of boundary value problem (57)–(59) and μ_j be two known coefficients of boundary conditions (59). Then we can find two coefficients h_j. Substituting these two values s_1 and s_2 in $\Delta(\mu)=0$, we obtain a system of two equations $\Delta(s_1)=0,\ \Delta(s_2)=0$ w.r.t the unknowns m_1 and m_2.

Example 1. *Let $\mu_1^2=-1.7479^2$, $\mu_2^2=-4.7467^2$ be the eigenvalues of boundary value problem (57)–(59) and $m_1=0.1$, $m_2=0.07$ be the coefficients of boundary conditions (59). Then the system of two equations $\Delta(\mu_1)=0,\ \Delta(\mu_2)=0$ w.r.t the unknowns h_1 and h_2 has two solutions: $h_1=-7$, $h_2=-2$ and $h_1=-2.0918$, $h_2=-6.3241$.*

Example 2. *Let $\mu_1^2=-1.7479^2$, $\mu_2^2=-4.7467^2$ be the eigenvalues of boundary value problem (57)–(59) and $h_1=-7$, $h_2=-2$ be the coefficients of boundary conditions (59). Then the system of two equations $\Delta(\mu_1)=0,\ \Delta(\mu_2)=0$ w.r.t the unknowns m_1 and m_2 has two solutions: $m_1=0.1$, $m_2=0.07$ and $m_1=1.7040$, $m_2=-0.12192$.*

Consider the solution of the third inverse problem for the boundary value problem (57)–(59). We have in the case $e(0,\mu)\neq 0$

$$\Delta(\mu)=\alpha(\mu)\left[\frac{c_1'(1,\mu)+H_1c_1(1,\mu)}{s_1'(1,\mu)+H_1s_1(1,\mu)}+\frac{c_2'(1,\mu)+H_2c_2(1,\mu)}{s_2'(1,\mu)+H_2s_2(1,\mu)}-\frac{e'(0,\mu)}{e(0,\mu)}\right].$$

It is known that $2n+1$ eigenvalues of the problems visible at infinity and (or) resonances of boundary value problem (57)–(59) are zeros of the function $\Delta(\mu)$.

If we know zeros of the function $\Delta(\mu)$ we can find boundary conditions (59). Let μ_k (k=1,2,3,4) be the zeros of the function $\Delta(\mu)$. Substituting these μ_k (k=1,2,3,4,5) in $\Delta(\mu)=0$, we obtain a system of four equations $\Delta(s_k)=0$(k=1,2,3,4,5) w.r.t the unknowns h_1, h_2, m_1 and m_2. Let us show that if we use five eigenvalues to reconstruct the boundary conditions, then the problem of identifying boundary conditions has two solutions.

Example 3. *Let $q(x_j)\equiv 0$; $\mu_1=-0.93575-0.34639\cdot i$, $\mu_2=-6.0292-0.48096\cdot i$, $\mu_3=-17.900-0.52894\cdot i$ and $s_4=-27.133-0.53887\cdot i$ be the zeros of the function $\Delta(\mu)$ (the square roots of the resonances of boundary value problem (57)–(59). Then the system of four equations $\Delta(\mu_k)=0$ (k=1,2,3,4) w.r.t the unknowns h_1, h_2, m_1 and m_2 has six solutions:*

$$h_1 = 1, \quad h_2 = 2, \quad m_1 = 0.1, \quad m_2 = 0.07;$$

$$h_1 = 2, \quad h_2 = 1, \quad m_1 = 0.07, \quad m_2 = 0.1;$$

$$h_1 = 0.73 - 0.028i,\ h_2 = 5.3 + 2.5i,\ m_1 = 0.24 - 0.18i,\ m_2 = 0.045 + 0.00065i;$$

$$h_1 = 1.0 + 0.027i,\ h_2 = 1.9 - 0.10i,\ m_1 = 0.075 - 0.0038i,\ m_2 = 0.092 + 0.0063i;$$

$$h_1 = 5.3 + 2.5i,\ h_2 = 0.73 - 0.028i,\ m_1 = 0.045 + 0.00066i,\ m_2 = 0.25 - 0.18i;$$

$$h_1 = 1.9 - 0.10i,\ h_2 = 1.0 + 0.027i,\ m_1 = 0.092 + 0.0063i,\ m_2 = 0.075 - 0.0039i;$$

Example 4. *Let* $q(x_j) \equiv 0$; $\mu_1 = -0.93575 - 0.34639 \cdot i$, $\mu_2 = -6.0292 - 0.48096 \cdot i$, $\mu_3 = -17.900 - 0.52894 \cdot i$ *and* $\mu_5 = -0.93575 - 0.34639 \cdot i$ *be the zeros of the function* $\Delta(\mu)$ *(the square roots of the resonances of boundary value problem (57)–(59). Then the system of four equations* $\Delta(\mu_k) = 0$ *(k=1,2,3,5) w.r.t the unknowns* h_1, h_2, m_1 *and* m_2 *has six solutions:*

$$h_1 = 1,\ h_2 = 2,\ m_1 = 0.1,\ m_2 = 0.07;$$

$$h_1 = 2,\ h_2 = 1,\ m_1 = 0.07,\ m_2 = 0.1;$$

$$h_1 = 0.88 - 0.015i,\ h_2 = 2.3 + 0.048i,\ m_1 = 0.064 - 0.0055i,\ m_2 = 0.11 + 0.10i;$$

$$h_1 = -6.4 + 7.1i,\ h_2 = 0.091 - 0.11i,\ m_1 = -0.081 - 0.0028i,\ m_2 = -0.16 - 0.039i;$$

$$h_1 = 0.091 - 0.11i,\ h_2 = -6.4 + 7.1i,\ m_1 = -0.16 - 0.039i,\ m_2 = -0.0081 - 0.0028i;$$

$$h_1 = 2.3 + 0.048i,\ h_2 = 0.88 - 0.015i,\ m_1 = 0.11 + 0.010i,\ m_2 = 0.064 - 0.0055i;$$

Example 5. *Let* $q(x_j) \equiv 0$; $\mu_1 = -0.93575 - 0.34639 \cdot i$, $\mu_2 = -6.0292 - 0.48096 \cdot i$, $\mu_3 = -17.900 - 0.52894 \cdot i$, $\mu_4 = -27.133 - 0.53887 \cdot i$ *and* $\mu_5 = -0.93575 - 0.34639 \cdot i$ *be the zeros of the function* $\Delta(\mu)$ *(the square roots of the resonances of boundary value problem (57)–(59)). We get the system of five equations* $\Delta(\mu_k) = 0$ *(k=1,2,3,4,5) w.r.t the unknowns* h_1*,* h_2*,* m_1 *and* m_2*. The solutions of this system are found as the intersection of the solutions from Examples 3 and 4. The intersection of the solutions from Examples 3 and 4 has two solutions:* $h_1 = 1$*,* $h_2 = 2$*,* $m_1 = 0.1$*,* $m_2 = 0.07$*;* $h_1 = 2$*,* $h_2 = 1$*,* $m_1 = 0.07$*,* $m_2 = 0.1$*;*

Thus, if $l_j = l$, then the boundary conditions are uniquely determined up to permutations on the dead-end vertices. The situation is similar in the general case, for the reconstruction of the boundary conditions of the problem (1)–(4). If $2n + 1$ resonances are used to reconstruct the boundary conditions (4) ($2n$ unknown coefficients), then the problem of identification of boundary conditions (4) has $n!$ solutions. The boundary conditions are uniquely determined up to permutations on the dead-end vertices. This follows from the fact that if in the equality (54) we rearrange the first n terms, then the equation remains the same.

If they also had $2n+1$ eigenvalues visible at infinity, then we could also restore the boundary conditions (4) up to permutations on the dead-end vertices.

Acknowledgments

The work was supported by Russian Foundation for Basic Research and the government of the Republic of Bashkortostan (projects 18-51-06002-Az_a, 18-01-00250-a, 17-41-020230-r_a, 17-41-020195-r_a).

References

[1] Z.F. Aksenova, A.M. Akhtyamov, *Acoustic diagnostic of concentrated mass at end of a string graph with elastic fastening at the ends*, Vestnik Bashkirskogo universiteta (Bulletin of Bashkir University), **19(1)**, 2014, 14-18 (in Russian).

[2] A.M. Akhtyamov, I. Trooshin, *Identification of boundary conditions for infinite star graph*, Modern Mathematics and its Applications, Papers of The International Scientific-Practical Conference, Part 1, Ufa, 2017, 6-10.

[3] M.I. Belishev, *Boundary spectral inverse problem on a class of graphs (trees) by the BC method*, Inverse Problems, **20**, 2004, 647-672.

[4] B.M. Brown, R.A. Weikard, *Borg-Levinson theorem for trees*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., **464(2062)**, 2005, 3231-3243.

[5] M.D. Faddeev, B.S. Pavlov, *Model of free electrons and the scattering problem*, Theor. Math. Phys., **55(2)**, 1983, 485-492.

[6] T. Kottos, U. Smilansky, *Quantum chaos on graphs*, Phys. Rev. Lett., **79**, 1997, 4794-4797.

[7] G. Berkolaiko, P. Kuchment, *Introduction to quantum graphs, Mathematical Surveys and Monographs*, **186**, Providence, RI, American Math. Society, 2013.

[8] P. Exner, P. Seba, *Free quantum motion on a branching graph*, Rep. Math. Phys., 1989, 7–26

[9] J.E. Langese, G. Leugering, J.P. Schmidt, *Modelling, analysis and control of dynamic elastic multi-link structures*, Birkhäuser, Boston, 1994.

[10] B.M. Levitan, *Inverse Sturm-Liouville problems*, Nauka, Moscow, 1984; English transl., VNU Sci.Press, Utrecht, 1987.

[11] V.A. Marchenko, *Sturm-Liouville operators and their applications*, Naukova Dumka, Kiev, 1977; English transl., Birkhäuser, 1986.

[12] V.A. Marchenko, K. Mochizuki, I. Trooshin, *Inverse scattering on a graph containing circle*, Analytic methods of analysis and differential equations: AMADE 2006 (Cambridge: Camb. Sci. Publ), 2008, 237-243.

[13] K. Mochizuki, I. Trooshin, *On the scattering on a loop-shaped graph*, Evolution Equations of Hyperbolic and Schrödinger Type, Progress in Mathematics, **301**, 2012, 227-245.

[14] K. Mochizuki, I. Trooshin, *Spectral problems and scattering on noncompact graphs containing finite rays*, J. Inverse Ill-Posed Probl., **23(1)**, 2015, 23-40.

[15] K. Mochizuki, I. Trooshin, *A stationary approach to the scattering on noncompact graphs containing finite rays*, Current Trends in Analysis and its Applications Proceedings of the 9th ISAAC Congress, Krakow 2013, Springer Proceedings in Mathematics and Statistic, 2015, 253-263.

[16] K. Mochizuki, I. Trooshin, *On Inverse Scattering on a Sun-Type Graph*, New Trends in Analysis and Interdisciplinary Applications, Trends in Mathematics Research Perspectives, Birkhauser, Springer, 2017, 307-313.

[17] Yu.V. Pokornyi, O.M. Penkin, V.L. Pryadiev, A.V. Borovskikh, K.P. Lazarev, S.A. Shabrov, *Differentsial'nye uravneniya na geometricheskikh*

grafakh [Differential equations on geometric graphs], Moscow: Fizmatlit, 2005, 272 p. (in Russian)

[18] Yu.V. Pokornyi, A.V. Borovskikh, *Differential equations on networks (geometric graphs)*, J. Math. Sci., **119(6)**, 2004, 691-718.

[19] Yu.V. Pokornyi, V. Pryadiev, *The qualitative Sturm-Liouville theory on spatial networks*, J. Math. Sci., **119(6)**, 2004, 788-835.

[20] A. Sobolev, M. Solomyak, *Schrödinger operator on homogeneous metric trees: spectrum in gaps*, Rev. Math. Phys., **14(5)**, 2002, 421-467.

[21] V.A. Yurko, *Inverse problems for Sturm-Liouville operators on graphs with a cycle*, Operators and Matrices, **2(4)**, 2008, 543-553.

[22] V.A. Yurko, *Uniqueness of recovering differential operators on hedgehog-type graphs*, Advances in Dynamical Systems and Applications, **4(2)**, 2009, 231-241.

[23] V.A. Yurko, *Inverse problems for Sturm-Liouville operators on bush-type graphs*, Inverse Problems, **25(10)**, 2009, 105008, 14 pp.

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Received 26 May 2018
Accepted 01 August 2018