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Solvability Conditions in Weighted Sobolev Type Spaces for one Class of Inverse Parabolic Operator-Differential Equations

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Abstract. In this paper, we obtain sufficient conditions for the well-posed and unique solvability in a weighted Sobolev space for a class of inverse parabolic operator-differential equations of third order. The main part of the equation under consideration has a multiple characteristic. We establish a connection between the solvability conditions and the values of the norms of intermediate derivatives operators. These norms are estimated with respect to the norm of the operator generated by the main part of considered equation. The obtained results show the role of the lower boundary of the spectrum of an abstract operator appearing in the main part of the equation.

Key Words and Phrases: operator-differential equation, weighted Sobolev space, regular solvability, self-adjoint operator, norms of intermediate derivative operators.

2010 Mathematics Subject Classifications: 35K25, 35K90, 47A63

1. Problem statement

Let H be a separable Hilbert space with scalar product (x, y), $x, y \in H$ and A be a self-adjoint positive-definite operator in H ($A = A^* \geq cE$, $c > 0$, E is a unit operator).

We denote by $L_2(R; H)$ $(R = (-\infty, +\infty))$ the space of measurable (see [1]) functions with values from H, endowed with the norm

$$\|g\|_{L_2(R;H)} = \left(\int_{-\infty}^{+\infty} \|g(t)\|^2 \, dt \right)^{\frac{1}{2}},$$

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and by $W_2^3(R;H)$ the space of functions with values from H such that $\frac{d^3v(t)}{dt^3}$, $A^3v(t) \in L_2(R;H)$, endowed with the norm

$$\|v\|_{W_2^3(R;H)} = \left(\left\| \frac{d^3v}{dt^3} \right\|^2_{L_2(R;H)} + \left\| A^3 v \right\|^2_{L_2(R;H)} \right)^{\frac{1}{2}}.$$

See [2, Ch.1] for more details about space $W_2^3(R;H)$.

Note that all derivatives are hereinafter understood in the sense of the theory of distributions; the operator A^γ is determined from the spectral decomposition of the operator A, i.e. $A^\gamma = \int_c^{+\infty} \sigma^\gamma dE_\sigma, \ \ \gamma \geq 0,$ where E_σ is the decomposition of unit of the operator A.

Let $-\infty < \kappa < +\infty$. For the functions $u(t)$, defined in R, with values from H, we introduce the following spaces with the weight $e^{-\frac{\kappa}{2}t}$:

$$L_{2,\kappa}(R;H) = \left\{ u(t) : \|u\|_{L_{2,\kappa}(R;H)} = \left(\int_{-\infty}^{+\infty} \|u(t)\|_H^2 \, e^{-\kappa t} dt \right)^{\frac{1}{2}} < +\infty \right\},$$

$$W_{2,\kappa}^3(R;H) = \left\{ u(t) : \|u\|_{W_{2,\kappa}^3(R;H)} = \right.$$

$$\left. \left(\int_{-\infty}^{+\infty} \left(\left\| \frac{d^3u(t)}{dt^3} \right\|_H^2 + \left\| A^3 u(t) \right\|_H^2 \right) e^{-\kappa t} dt \right)^{\frac{1}{2}} < +\infty \right\}.$$

It is evident that for $\kappa = 0$ we will have the spaces $L_{2,0}(R;H) = L_2(R;H)$ and $W_{2,0}^3(R;H) = W_2^3(R;H)$.

Hereinafter, by $L(X,Y)$ we mean the set of linear bounded operators from the Hilbert space X to another Hilbert space Y. If $Y = X$, then we write $L(X)$ instead of $L(X,Y)$. By $\sigma(A)$ we mean the spectrum of the operator A.

Consider the operator-differential equation

$$\left(-\frac{d}{dt} + A \right)^3 u(t) + A_1 \frac{d^2u(t)}{dt^2} + A_2 \frac{du(t)}{dt} + A_3 u(t) = f(t), t \in R, \tag{1}$$

where $A = A^* \geq cE$, $c > 0$, A_1, A_2, A_3 are linear, and generally speaking, unbounded operators, $f(t) \in L_{2,\kappa}(R;H)$, $u(t) \in W_{2,\kappa}^3(R;H)$.

The equation (1) for $A_1 = A_2 = A_3 = 0$ has a multiple characteristic and, according to the classification carried out in [3], it belongs to the class of inverse parabolic operator-differential equations. The class of inverse parabolic equations is dual to the class of parabolic equations. Such equations often appear in various fields of natural science; for example, they characterize the problems of diffusion or heat conduction in a viscoelastic medium.

Definition 1. *If for* $f(t) \in L_{2,\kappa}(R;H)$ *there exists a vector-function* $u(t) \in W^3_{2,\kappa}(R;H)$ *that satisfies equation* (1) *almost everywhere, then we will call it a regular solution of equation* (1)*.*

Definition 2. *If for any* $f(t) \in L_{2,\kappa}(R;H)$ *there exists a regular solution of equation* (1) *and the inequality*

$$\|u\|_{W^3_{2,\kappa}(R;H)} \leq const\, \|f\|_{L_{2,\kappa}(R;H)}$$

holds, then equation (1) *is said to be regularly solvable.*

In this paper, we find conditions for regular solvability of the equation (1).

In the mathematical literature, there is a large amount of research dedicated to operator-differential equations. The works by E.Hille, K.Iosida, T.Kato, S.Agmon, L.Nirinberg, and Z.I.Khalilov laid the foundation for the theory of these equations in the mid-20th Century. Later, a whole series of fundamental results were obtained both in the direction of solvability issues for operator-differential equations and in the field of spectral problems of polynomial operator pencils related to these equations. Among these works, we should mention, for example, the papers by M.G.Gasymov, A.G.Kostyuchenko, G.V.Radzievsky, A.A.Shkalikov, S.S.Mirzoev, A.R.Aliev (see, for example, [4]-[15] and references therein). But in the majority of these papers, the considered operator-differential equations do not have a multiple characteristic. As noted above, the main part of equation (1) has a multiple characteristic. Over the past 10 years, the works [16]-[24] have appeared, in which the issues of regular and normal solvability of operator-differential equations with a multiple characteristic have been investigated. It should be stressed here that the equations studied in those papers belong to the class of quasi-elliptic operator-differential equations. The main part of equation (1) is the inverse parabolic equation. Despite the fact that the issues of regular solvability of parabolic operator-differential equations with a multiple characteristic are relatively little studied, a series of works [25]-[27] have appeared recently in this field, which, in conjunction with this paper, can further serve as an impetus for a thorough study of operator-differential equations of parabolic and inverse parabolic types with a multiple characteristic.

2. Isomorphism theorems

We first investigate equation (1) for $A_1 = A_2 = A_3 = 0$.

Denote by P_0 an operator from the space $W^3_{2,\kappa}(R;H)$ to the space $L_{2,\kappa}(R;H)$ defined as follows:

$$\mathrm{P}_0 u(t) \equiv \left(-\frac{d}{dt} + A\right)^3 u(t)\,, u(t) \in W^3_2(R;H)\,.$$

We have the following lemma.

Lemma 1. *Let* $A = A^* \geq cE$, $c > 0$. *Then the operator* P_0 *is bounded from the space* $W_{2,\kappa}^3(R;H)$ *to the space* $L_{2,\kappa}(R;H)$.

Proof. Indeed, taking into account the Cauchy-Schwartz inequality and the Young inequality, for $u\,(t) \in W_{2,\kappa}^3\,(R;H)$, we have:

$$-Re\left(\frac{d^3u}{dt^3},\, A\frac{d^2u}{dt^2}\right)_{L_{2,\kappa}(R;H)} \leq \left\|\frac{d^3u}{dt^3}\right\|_{L_{2,\kappa}(R;H)} \left\|A\frac{d^2u}{dt^2}\right\|_{L_{2,\kappa}(R;H)} \leq$$

$$\leq \frac{1}{2}\left\|\frac{d^3u}{dt^3}\right\|^2_{L_{2,\kappa}(R;H)} + \frac{1}{2}\left\|A\frac{d^2u}{dt^2}\right\|^2_{L_{2,\kappa}(R;H)}, \tag{2}$$

$$Re\left(\frac{d^3u}{dt^3},\, A^2\frac{du}{dt}\right)_{L_{2,\kappa}(R;H)} \leq \left\|\frac{d^3u}{dt^3}\right\|_{L_{2,\kappa}(R;H)} \left\|A^2\frac{du}{dt}\right\|_{L_{2,\kappa}(R;H)} \leq$$

$$\leq \frac{1}{2}\left\|\frac{d^3u}{dt^3}\right\|^2_{L_{2,\kappa}(R;H)} + \frac{1}{2}\left\|A^2\frac{du}{dt}\right\|^2_{L_{2,\kappa}(R;H)}, \tag{3}$$

$$-Re\left(\frac{d^3u}{dt^3},\, A^3u\right)_{L_{2,\kappa}(R;H)} \leq \left\|\frac{d^3u}{dt^3}\right\|_{L_{2,\kappa}(R;H)} \left\|A^3u\right\|_{L_{2,\kappa}(R;H)} \leq$$

$$\leq \frac{1}{2}\left\|\frac{d^3u}{dt^3}\right\|^2_{L_{2,\kappa}(R;H)} + \frac{1}{2}\left\|A^3u\right\|^2_{L_{2,\kappa}(R;H)}, \tag{4}$$

$$-Re\left(A\frac{d^2u}{dt^2},\, A^2\frac{du}{dt}\right)_{L_{2,\kappa}(R;H)} \leq \left\|A\frac{d^2u}{dt^2}\right\|_{L_{2,\kappa}(R;H)} \left\|A^2\frac{du}{dt}\right\|_{L_{2,\kappa}(R;H)} \leq$$

$$\leq \frac{1}{2}\left\|A\frac{d^2u}{dt^2}\right\|^2_{L_{2,\kappa}(R;H)} + \frac{1}{2}\left\|A^2\frac{du}{dt}\right\|^2_{L_{2,\kappa}(R;H)}, \tag{5}$$

$$Re\left(A\frac{d^2u}{dt^2},\, A^3u\right)_{L_{2,\kappa}(R;H)} \leq \left\|A\frac{d^2u}{dt^2}\right\|_{L_{2,\kappa}(R;H)} \left\|A^3u\right\|_{L_{2,\kappa}(R;H)} \leq$$

$$\leq \frac{1}{2}\left\|A\frac{d^2u}{dt^2}\right\|^2_{L_{2,\kappa}(R;H)} + \frac{1}{2}\left\|A^3u\right\|^2_{L_{2,\kappa}(R;H)}, \tag{6}$$

$$-Re\left(A^2\frac{du}{dt},\, A^3u\right)_{L_{2,\kappa}(R;H)} \leq \left\|A^2\frac{du}{dt}\right\|_{L_{2,\kappa}(R;H)} \left\|A^3u\right\|_{L_{2,\kappa}(R;H)} \leq$$

$$\leq \frac{1}{2}\left\|A^2\frac{du}{dt}\right\|^2_{L_{2,\kappa}(R;H)} + \frac{1}{2}\left\|A^3u\right\|^2_{L_{2,\kappa}(R;H)}. \tag{7}$$

Then, taking into account inequalities (2)-(7) and the intermediate derivatives theorem [2, Ch.1], we obtain:

$$\|\mathrm{P}_0 u\|^2_{L_{2,\kappa}(R;H)} = \left\|\frac{d^3u}{dt^3}\right\|^2_{L_{2,\kappa}(R;H)} + 9\left\|A\frac{d^2u}{dt^2}\right\|^2_{L_{2,\kappa}(R;H)} + 9\left\|A^2\frac{du}{dt}\right\|^2_{L_{2,\kappa}(R;H)} +$$

$$+\left\|A^3u\right\|^2_{L_{2,\kappa}(R;H)} - 6Re\left(\frac{d^3u}{dt^3},\, A\frac{d^2u}{dt^2}\right)_{L_{2,\kappa}(R;H)} +$$

$$+6Re\left(\frac{d^3u}{dt^3},\, A^2\frac{du}{dt}\right)_{L_{2,\kappa}(R;H)} - 2Re\left(\frac{d^3u}{dt^3},\, A^3u\right)_{L_{2,\kappa}(R;H)} -$$

$$-18Re\left(A\frac{d^2u}{dt^2},\, A^2\frac{du}{dt}\right)_{L_{2,\kappa}(R;H)} + 6Re\left(A\frac{d^2u}{dt^2},\, A^3u\right)_{L_{2,\kappa}(R;H)} -$$

$$-6Re\left(A^2\frac{du}{dt},\, A^3u\right)_{L_{2,\kappa}(R;H)} \le 8\left\|\frac{d^3u}{dt^3}\right\|^2_{L_{2,\kappa}(R;H)} + 24\left\|A\frac{d^2u}{dt^2}\right\|^2_{L_{2,\kappa}(R;H)} +$$

$$+24\left\|A^2\frac{du}{dt}\right\|^2_{L_{2,\kappa}(R;H)} + 8\left\|A^3u\right\|^2_{L_{2,\kappa}(R;H)} = 8\Big(\|u\|^2_{W^3_{2,\kappa}(R;H)} +$$

$$+3\left\|A\frac{d^2u}{dt^2}\right\|^2_{L_{2,\kappa}(R;H)} + 3\left\|A^2\frac{du}{dt}\right\|^2_{L_{2,\kappa}(R;H)}\Big) \le const\,\|u\|^2_{W^3_{2,\kappa}(R;H)},$$

i.e.

$$\|\mathrm{P}_0 u\|_{L_{2,\kappa}(R;H)} \le const\,\|u\|_{W^3_{2,\kappa}(R;H)}. \quad \blacktriangleleft$$

We now turn to the issue about isomorphism of the operator P_0.

We have the following theorem.

Theorem 1. *Let A be a self-adjoint positive-definite operator with the lower bound of the spectrum λ_0 ($A = A^* \ge \lambda_0 E$, $\lambda_0 > 0$) and $\kappa < 2\lambda_0$. Then the operator P_0 is an isomorphism between the spaces $W^3_{2,\kappa}(R;H)$ and $L_{2,\kappa}(R;H)$.*

Proof. In the equation

$$\mathrm{P}_0 u\,(t) = f\,(t)\,, \tag{8}$$

$u(t) \in W^3_{2,\kappa}(R;H)$, $f(t) \in L_{2,\kappa}(R;H)$, we make a substitution $u(t) = v(t)e^{\frac{\kappa}{2}t}$. Then, $v(t) = u(t)e^{-\frac{\kappa}{2}t} \in W^3_2(R;H)$. As

$$\left(-\frac{d}{dt} + A\right)^3 u(t) = e^{\frac{\kappa}{2}t}\left(-\frac{d}{dt} - \frac{\kappa}{2} + A\right)^3 v(t) = f(t),$$

we obtain

$$\left(-\frac{d}{dt}-\frac{\kappa}{2}+A\right)^3 v(t)=f(t)e^{-\frac{\kappa}{2}t}. \tag{9}$$

By virtue of the fact that $g(t)=f(t)e^{-\frac{\kappa}{2}t}\in L_2(R;H)$, equation (9) can be written as

$$\left(-\frac{d}{dt}-\frac{\kappa}{2}+A\right)^3 v(t)=g(t) \tag{10}$$

in the space $L_2(R;H)$, where $v(t)\in W_2^3(R;H)$, $g(t)\in L_2(R;H)$.

Denote

$$\mathrm{P}_{0,\kappa}v(t)=\left(-\frac{d}{dt}-\frac{\kappa}{2}+A\right)^3 v(t), v(t)\in W_2^3(R;H).$$

Then equation (10) can be written as $\mathrm{P}_{0,\kappa}v(t)=g(t)$, where $v(t)\in W_2^3(R;H)$, $g(t)\in L_2(R;H)$. To solve the last equation we use the Fourier transform:

$$\left(-\left(i\xi+\frac{\kappa}{2}\right)E+A\right)^3\hat{v}(\xi)=\hat{g}(\xi), \tag{11}$$

where $\hat{v}(\xi)$, $\hat{g}(\xi)$ are Fourier transforms of the vector-functions $v(t)$, $g(t)$, respectively. Let us prove that for $\kappa<2\lambda_0$, the operator sheaf

$$\mathrm{P}_{0,\kappa}(i\xi;A)=\left(-\left(i\xi+\frac{\kappa}{2}\right)E+A\right)^3 \tag{12}$$

is invertible. Let $\lambda\in\sigma(A)$ $(\lambda\geq\lambda_0)$. Then the characteristic polynomial (12) has the form

$$\mathrm{P}_{0,\kappa}(i\xi;\lambda)=\left(-i\xi-\frac{\kappa}{2}+\lambda\right)^3.$$

From here we have

$$|\mathrm{P}_{0,\kappa}(i\xi;\lambda)|=\left|\left(-i\xi-\frac{\kappa}{2}+\lambda\right)^3\right|=\left(\left(\lambda-\frac{\kappa}{2}\right)^2+\xi^2\right)^{\frac{3}{2}}\geq$$

$$\geq\left(\lambda-\frac{\kappa}{2}\right)^3\geq\left(\lambda_0-\frac{\kappa}{2}\right)^3>0, \xi\in R,$$

i.e. from the spectral decomposition of the operator A it follows that the operator sheaf $\mathrm{P}_{0,\kappa}(i\xi;A)$ is reversible for $\kappa<2\lambda_0$. Then from (11) we can find $\hat{v}(\xi)$:

$$\hat{v}(\xi)=\left(-\left(i\xi+\frac{\kappa}{2}\right)E+A\right)^{-3}\hat{g}(\xi).$$

Therefore,

$$v(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left(-\left(i\xi+\frac{\kappa}{2}\right)E+A\right)^{-3}\hat{g}(\xi)e^{i\xi t}d\xi.$$

It is clear that $v(t)$ satisfies equation (10) almost everywhere.

Now we'll try to prove that $v(t)\in W_2^3(R;H)$. According to the well-known Plancherel theorem, it suffices to prove that $A^3\hat{v}(\xi)\in L_2(R;H)$ and $-i\xi^3\hat{v}(\xi)\in L_2(R;H)$. It is evident that

$$\|v\|^2_{W_2^3(R;H)}=\left\|\frac{d^3v}{dt^3}\right\|^2_{L_2(R;H)}+\left\|A^3v\right\|^2_{L_2(R;H)}=$$

$$=\left\|-i\xi^3\hat{v}(\xi)\right\|^2_{L_2(R;H)}+\left\|A^3\hat{v}(\xi)\right\|^2_{L_2(R;H)}.$$

As

$$\left\|A^3\hat{v}(\xi)\right\|_{L_2(R;H)}=\left\|A^3\left(-\left(i\xi+\frac{\kappa}{2}\right)E+A\right)^{-3}\hat{g}(\xi)\right\|_{L_2(R;H)}\le$$

$$\le\sup_{\xi\in R}\left\|A^3\left(-\left(i\xi+\frac{\kappa}{2}\right)E+A\right)^{-3}\right\|_{H\to H}\|\hat{g}(\xi)\|_{L_2(R;H)},$$

we estimate the norm $\left\|A^3\left(-\left(i\xi+\frac{\kappa}{2}\right)E+A\right)^{-3}\right\|_{H\to H}$ for $\xi\in R$. It follows from the spectral theory of self-adjoint operators that

$$\left\|A^3\left(-\left(i\xi+\frac{\kappa}{2}\right)E+A\right)^{-3}\right\|_{H\to H}=\sup_{\lambda\in\sigma(A)}\left|\lambda^3\left(-i\xi-\frac{\kappa}{2}+\lambda\right)^{-3}\right|=$$

$$=\sup_{\lambda\in\sigma(A)}\frac{\lambda^3}{\left(\left(\lambda-\frac{\kappa}{2}\right)^2+\xi^2\right)^{\frac{3}{2}}}\le$$

$$\le\sup_{\lambda\in\sigma(A)}\frac{\lambda^3}{\left(\lambda-\frac{\kappa}{2}\right)^3}\le\max\left\{\frac{\lambda_0^3}{\left(\lambda_0-\frac{\kappa}{2}\right)^3},\,1\right\}.$$

Then

$$\left\|A^3\left(-\left(i\xi+\frac{\kappa}{2}\right)E+A\right)^{-3}\hat{g}(\xi)\right\|_{L_2(R;H)}\le\max\left\{\frac{\lambda_0^3}{\left(\lambda_0-\frac{\kappa}{2}\right)^3},\,1\right\}\|\hat{g}(\xi)\|_{L_2(R;H)}.$$

Similarly, we have

$$\left\|-i\xi^3\hat{v}(\xi)\right\|_{L_2(R;H)}=\left\|-i\xi^3\left(-\left(i\xi+\frac{\kappa}{2}\right)E+A\right)^{-3}\hat{g}(\xi)\right\|_{L_2(R;H)}\le$$

$$\leq \sup_{\xi \in R} \left\| -i\xi^3 \left(-\left(i\xi + \frac{\kappa}{2} \right) E + A \right)^{-3} \right\|_{H \to H} \|\hat{g}(\xi)\|_{L_2(R;H)} .$$

Hence for $\xi \in R$ and $\kappa < 2\lambda_0$, we obtain:

$$\left\| -i\xi^3 \left(-\left(i\xi + \frac{\kappa}{2} \right) E + A \right)^{-3} \right\|_{H \to H} = \sup_{\lambda \in \sigma(A)} \left| -i\xi^3 \left(-i\xi - \frac{\kappa}{2} + \lambda \right)^{-3} \right| =$$

$$= \sup_{\lambda \in \sigma(A)} \frac{|\xi|^3}{\left(\left(\lambda - \frac{\kappa}{2} \right)^2 + \xi^2 \right)^{\frac{3}{2}}} \leq \frac{|\xi|^3}{\left(\left(\lambda_0 - \frac{\kappa}{2} \right)^2 + \xi^2 \right)^{\frac{3}{2}}} \leq 1.$$

Then

$$\left\| -i\xi^3 \hat{v}(\xi) \right\|_{L_2(R;H)} \leq \|\hat{g}(\xi)\|_{L_2(R;H)} .$$

So $v(t) \in W_2^3(R;H)$.

It is obvious that the vector-function $v(t)e^{\frac{\kappa}{2}t} \in W_{2,\kappa}^3(R;H)$ is a regular solution of equation (8).

We also note the fact that it is obvious that the equation $\mathrm{P}_0 u(t) = 0$ has only a trivial solution from space $W_{2,\kappa}^3(R;H)$.

Thus, we have found out that the operator $\mathrm{P}_0 : W_{2,\kappa}^3(R;H) \to L_{2,\kappa}(R;H)$ is one-to-one and bounded by Lemma 1. Then, by the Banach theorem on the inverse operator, it follows that the operator $\mathrm{P}_0^{-1} : L_{2,\kappa}(R;H) \to W_{2,\kappa}^3(R;H)$ is bounded. Therefore, P_0 is an isomorphism between spaces $W_{2,\kappa}^3(R;H)$ and $L_{2,\kappa}(R;H)$. This proves the theorem. ◀

Corollary 1. *For $\kappa < 2\lambda_0$, the norms $\|\mathrm{P}_0 u\|_{L_{2,\kappa}(R;H)}$ and $\|u\|_{W_{2,\kappa}^3(R;H)}$ are equivalent in the space $W_{2,\kappa}^3(R;H)$.*

Remark 1. *For $\kappa = 2\lambda_0$, the operator P_0 is not invertible.*

Now we study the equation (1) for $A_j \neq 0$, $j = 1, 2, 3$.

Let's denote by P_1 an operator from the space $W_{2,\kappa}^3(R;H)$ to the space $L_{2,\kappa}(R;H)$ defined as follows:

$$\mathrm{P}_1 u(t) \equiv A_1 \frac{d^2 u(t)}{dt^2} + A_2 \frac{du(t)}{dt} + A_3 u(t), u(t) \in W_{2,\kappa}^3(R;H).$$

We have the following lemma.

Lemma 2. *Let $A = A^* \geq cE$, $c > 0$, and the operators $A_j A^{-j} \in L(H)$, $j = 1, 2, 3$. Then the operator P_1 is bounded from the space $W_{2,\kappa}^3(R;H)$ to the space $L_{2,\kappa}(R;H)$.*

Proof. Let $u(t) \in W_{2,\kappa}^{3}(R;H)$. Taking into account the conditions of the lemma and the theorem on intermediate derivatives [2, Ch.1], we have:

$$\|\mathrm{P}_1 u\|_{L_{2,\kappa}(R;H)} = \left\| A_1 \frac{d^2 u}{dt^2} + A_2 \frac{du}{dt} + A_3 u \right\|_{L_{2,\kappa}(R;H)} \leq$$

$$\leq \left\| A_1 \frac{d^2 u}{dt^2} \right\|_{L_{2,\kappa}(R;H)} + \left\| A_2 \frac{du}{dt} \right\|_{L_{2,\kappa}(R;H)} + \|A_3 u\|_{L_{2,\kappa}(R;H)} \leq$$

$$\leq \left\| A_1 A^{-1} \right\|_{H\to H} \left\| A \frac{d^2 u}{dt^2} \right\|_{L_{2,\kappa}(R;H)} + \left\| A_2 A^{-2} \right\|_{H\to H} \left\| A^2 \frac{du}{dt} \right\|_{L_{2,\kappa}(R;H)} +$$

$$+ \left\| A_3 A^{-3} \right\|_{H\to H} \left\| A^3 u \right\|_{L_{2,\kappa}(R;H)} \leq const \, \|u\|_{W_{2,\kappa}^{3}(R;H)},$$

i.e.

$$\|\mathrm{P}_1 u\|_{L_{2,\kappa}(R;H)} \leq const \, \|u\|_{W_{2,\kappa}^{3}(R;H)}. \quad \blacktriangleleft$$

Let's denote by P an operator from $W_{2,\kappa}^{3}(R;H)$ to $L_{2,\kappa}(R;H)$ defined as follows:

$$\mathrm{P}u(t) = \mathrm{P}_0 u(t) + \mathrm{P}_1 u(t), \, u(t) \in W_{2,\kappa}^{3}(R;H).$$

The following lemma is valid, for the proof of which we use Lemmas 1 and 2.

Lemma 3. *Let $A = A^* \geq cE$, $c > 0$, and the operators $A_j A^{-j} \in L(H)$, $j = 1,2,3$. Then the operator* P *is a bounded operator from the space $W_{2,\kappa}^{3}(R;H)$ to the space $L_{2,\kappa}(R;H)$.*

So, we can formulate the conditional theorem on solvability of equation (1).

Theorem 2. *Let $A = A^* \geq \lambda_0 E$, $\lambda_0 > 0$, $\kappa < 2\lambda_0$, $A_j A^{-j} \in L(H)$, $j = 1,2,3$, and the inequality*

$$N_1 \left\| A_1 A^{-1} \right\|_{H\to H} + N_2 \left\| A_2 A^{-2} \right\|_{H\to H} + N_3 \left\| A_3 A^{-3} \right\|_{H\to H} < 1,$$

hold, where

$$N_j = \sup_{0 \neq u \in W_{2,\kappa}^{3}(R;H)} \left(\left\| A^j \frac{d^{3-j} u}{dt^{3-j}} \right\| \|\mathrm{P}_0 u\|_{L_{2,\kappa}(R;H)}^{-1} \right), j = 1,2,3.$$

Then equation (1) *is regularly solvable.*

Proof. First, we rewrite equation (1) as an operator equation

$$\mathrm{P}u\,(t) = \mathrm{P}_0 u\,(t) + \mathrm{P}_1 u\,(t) = f(t), \tag{13}$$

where $f(t) \in L_{2,\kappa}(R;H)$, $u\,(t) \in W^3_{2,\kappa}\,(R;H)$.

By virtue of Theorem 1, equation (8) is regularly solvable. Let's make a substitution $\mathrm{P}_0 u(t) = w(t)$. Then equation (13) can be rewritten as:

$$\left(E + \mathrm{P}_1\,\mathrm{P}_0^{-1}\right) w(t) \,= f(t).$$

On the other hand, for every $w(t) \in L_{2,\kappa}(R;H)$, we have:

$$\left\|\mathrm{P}_1\,\mathrm{P}_0^{-1} w\right\|_{L_{2,\kappa}(R;H)} = \left\|\mathrm{P}_1\,u\right\|_{L_{2,\kappa}(R;H)} \le \left\|A_1 A^{-1}\right\|_{H\to H} \left\|A\frac{d^2u}{dt^2}\right\|_{L_{2,\kappa}(R;H)} +$$

$$+\left\|A_2 A^{-2}\right\|_{H\to H} \left\|A^2\frac{du}{dt}\right\|_{L_{2,\kappa}(R;H)} + \left\|A_3 A^{-3}\right\|_{H\to H} \left\|A^3 u\right\|_{L_{2,\kappa}(R;H)} \le$$

$$\le \left\|A_1 A^{-1}\right\|_{H\to H} N_1 \left\|\mathrm{P}_0 u\right\|_{L_{2,\kappa}(R;H)} + \left\|A_2 A^{-2}\right\|_{H\to H} N_2 \left\|\mathrm{P}_0 u\right\|_{L_{2,\kappa}(R;H)} +$$

$$+\left\|A_3 A^{-3}\right\|_{H\to H} N_3 \left\|\mathrm{P}_0 u\right\|_{L_{2,\kappa}(R;H)} =$$

$$= \left(N_1 \left\|A_1 A^{-1}\right\|_{H\to H} + N_2 \left\|A_2 A^{-2}\right\|_{H\to H} + N_3 \left\|A_3 A^{-3}\right\|_{H\to H}\right) \|w\|_{L_{2,\kappa}(R;H)}\,.$$

But, as

$$N_1 \left\|A_1 A^{-1}\right\|_{H\to H} + N_2 \left\|A_2 A^{-2}\right\|_{H\to H} + N_3 \left\|A_3 A^{-3}\right\|_{H\to H} < 1,$$

the operator $E + \mathrm{P}_1\,\mathrm{P}_0^{-1}$ is invertible in the space $L_{2,\kappa}(R;H)$. Therefore, $u(t)$ can be defined by the formula

$$u(t) = \mathrm{P}_0^{-1}\left(E + \mathrm{P}_1 \mathrm{P}_0^{-1}\right)^{-1} f(t),$$

with

$$\|u\|_{W^3_{2,\kappa}(R;H)} \le$$

$$\le \left\|\mathrm{P}_0^{-1}\right\|_{L_{2,\kappa}(R;H)\to W^3_{2,\kappa}(R;H)} \left\|\left(E + \mathrm{P}_1 \mathrm{P}_0^{-1}\right)^{-1}\right\|_{L_{2,\kappa}(R;H)\to L_{2,\kappa}(R;H)} \|f\|_{L_{2,\kappa}(R;H)} \le$$

$$\le const\,\|f\|_{L_{2,\kappa}(R;H)}\,.$$

This proves the theorem. ◄

Corollary 2. *Under the conditions of Theorem 2, the operator* P *is an isomorphism between the spaces* $W^3_{2,\kappa}(R;H)$ *and* $L_{2,\kappa}(R;H)$.

3. Estimation of numbers N_j, $j = 1, 2, 3$

From Theorem 2 it becomes clear that there arises a problem of the exact value or estimation of numbers N_j, $j = 1, 2, 3$.

First, we estimate the norms of intermediate derivative operators

$$A^j \frac{d^{3-j}}{dt^{3-j}} : W_{2,\kappa}^3(R; H) \to L_{2,\kappa}(R; H), j = 1, 2, 3,$$

with respect to $\|\mathrm{P}_0 u\|_{L_{2,\kappa}(R;H)}$ by virtue of Corollary 1 and the fact that these operators are continuous [2].

Theorem 3. *Let $A = A^* \geq \lambda_0 E$, $\lambda_0 > 0$, $\kappa < 2\lambda_0$. Then for every $u(t) \in W_{2,\kappa}^3(R; H)$, the following inequalities hold:*

$$\left\| A^j \frac{d^{3-j} u}{dt^{3-j}} \right\|_{L_{2,\kappa}(R;H)} \leq c_j(\kappa; \lambda_0) \|\mathrm{P}_0 u\|_{L_{2,\kappa}(R;H)}, j = 1, 2, 3, \tag{14}$$

where

$$c_1(\kappa; \lambda_0) = \begin{cases} \frac{2}{3^{3/2}\left(1 - \frac{\kappa}{\lambda_0}\right)^{1/2}}, & \kappa < \lambda_0, \\ \frac{2\lambda_0 \kappa^2}{(2\lambda_0 - \kappa)^3}, & \lambda_0 \leq \kappa < 2\lambda_0, \end{cases} \quad c_2(\kappa; \lambda_0) = \begin{cases} \frac{2}{3^{3/2}\left(1 - \frac{\kappa}{\lambda_0}\right)}, & \kappa < \lambda_0, \\ \frac{4\lambda_0^2 |\kappa|}{(2\lambda_0 - \kappa)^3}, & \lambda_0 \leq \kappa < 2\lambda_0, \end{cases}$$

$$c_3(\kappa; \lambda_0) = \left(1 - \frac{\kappa}{2\lambda_0}\right)^{-3}.$$

Proof. Note that to prove inequalities (14), it suffices to estimate the norms

$$\left\| A^j \left(\frac{d}{dt} + \frac{\kappa}{2}\right)^{3-j} v \right\|_{L_2(R;H)}, j = 1, 2, 3,$$

with respect to $\|\mathrm{P}_{0,\kappa} v\|_{L_2(R;H)}$, since the mapping $v(t) \to u(t) e^{-\frac{\kappa}{2} t}$ is an isomorphism between spaces $W_2^3(R; H)$ and $W_{2,\kappa}^3(R; H)$.

Replacing $\mathrm{P}_{0,\kappa} v(t) = g(t)$ and applying the Fourier transform, we have

$$\left\| A^j \left(i\xi + \frac{\kappa}{2}\right)^{3-j} \mathrm{P}_{0,\kappa}^{-1}(i\xi; A) \hat{g}(\xi) \right\|_{L_2(R;H)} \leq$$

$$\leq \sup_{\xi \in R} \left\| A^j \left(i\xi + \frac{\kappa}{2}\right)^{3-j} \left(-\left(i\xi + \frac{\kappa}{2}\right) E + A\right)^{-3} \right\|_{H \to H} \|\hat{g}(\xi)\|_{L_2(R;H)}, j = 1, 2, 3. \tag{15}$$

Then for $\xi \in R$ and $\kappa < 2\lambda_0$ it is necessary to estimate the norms

$$\left\| A^j \left(i\xi + \frac{\kappa}{2}\right)^{3-j} \left(-\left(i\xi + \frac{\kappa}{2}\right)E + A\right)^{-3}\right\|_{H\to H} = \sup_{\lambda\in\sigma(A)} \left| \frac{\lambda^j \left(i\xi + \frac{\kappa}{2}\right)^{3-j}}{\left(-i\xi - \frac{\kappa}{2} + \lambda\right)^3}\right| =$$

$$= \sup_{\lambda\in\sigma(A)} \frac{\lambda^j \left(\xi^2 + \frac{\kappa^2}{4}\right)^{\frac{3-j}{2}}}{\left(\left(\lambda - \frac{\kappa}{2}\right)^2 + \xi^2\right)^{3/2}}, j = 1, 2, 3.$$

First consider the case $j = 1$. Solving the extremum problem, we have:

$$\sup_{\lambda\in\sigma(A)} \frac{\lambda\left(\xi^2 + \frac{\kappa^2}{4}\right)}{\left(\left(\lambda - \frac{\kappa}{2}\right)^2 + \xi^2\right)^{3/2}} \le \sup_{\lambda\ge\lambda_0,\, \xi\in R} \frac{\frac{\xi^2}{\lambda^2} + \frac{\kappa^2}{4\lambda_0^2}}{\left(\left(1 - \frac{\kappa}{2\lambda_0}\right)^2 + \frac{\xi^2}{\lambda^2}\right)^{3/2}} =$$

$$= \sup_{\frac{\xi^2}{\lambda^2}\ge 0} \frac{\frac{\xi^2}{\lambda^2} + \frac{\kappa^2}{4\lambda_0^2}}{\left(\left(1 - \frac{\kappa}{2\lambda_0}\right)^2 + \frac{\xi^2}{\lambda^2}\right)^{3/2}} = c_1(\kappa;\lambda_0),$$

where $c_1(\kappa;\lambda_0) = \frac{2}{3^{3/2}\left(1-\frac{\kappa}{\lambda_0}\right)^{1/2}}$, if $\kappa < \lambda_0$ and $c_1(\kappa;\lambda_0) = \frac{2\lambda_0\kappa^2}{(2\lambda_0-\kappa)^3}$, if $\lambda_0 \le \kappa < 2\lambda_0$.

For $j = 2$ we obtain:

$$\sup_{\lambda\in\sigma(A)} \frac{\lambda^2\left(\xi^2 + \frac{\kappa^2}{4}\right)^{1/2}}{\left(\left(\lambda - \frac{\kappa}{2}\right)^2 + \xi^2\right)^{3/2}} \le \sup_{\lambda\ge\lambda_0,\, \xi\in R} \frac{\left(\frac{\xi^2}{\lambda^2} + \frac{\kappa^2}{4\lambda_0^2}\right)^{1/2}}{\left(\left(1 - \frac{\kappa}{2\lambda_0}\right)^2 + \frac{\xi^2}{\lambda^2}\right)^{3/2}} =$$

$$= \sup_{\frac{\xi^2}{\lambda^2}\ge 0} \frac{\left(\frac{\xi^2}{\lambda^2} + \frac{\kappa^2}{4\lambda_0^2}\right)^{1/2}}{\left(\left(1 - \frac{\kappa}{2\lambda_0}\right)^2 + \frac{\xi^2}{\lambda^2}\right)^{3/2}} = c_2(\kappa;\lambda_0),$$

where $c_2(\kappa;\lambda_0) = \frac{2}{3^{3/2}\left(1-\frac{\kappa}{\lambda_0}\right)}$, if $\kappa < \lambda_0$ and $c_2(\kappa;\lambda_0) = \frac{4\lambda_0^2|\kappa|}{(2\lambda_0-\kappa)^3}$, if $\lambda_0 \le \kappa < 2\lambda_0$.

It is obvious that in the case $j = 3$ we have:

$$\sup_{\lambda\in\sigma(A)} \frac{\lambda^3}{\left(\left(\lambda - \frac{\kappa}{2}\right)^2 + \xi^2\right)^{3/2}} \le \sup_{\lambda\in\sigma(A)} \frac{\lambda^3}{\left(\lambda - \frac{\kappa}{2}\right)^3} \le \left(1 - \frac{\kappa}{2\lambda_0}\right)^{-3} = c_3(\kappa;\lambda_0).$$

Further, taking the obtained estimates into account in inequalities (15), we have

$$\left\| A^j \left(i\xi + \frac{\kappa}{2} \right)^{3-j} \mathrm{P}_{0,\kappa}^{-1}(i\xi; A)\hat{g}(\xi) \right\|_{L_2(R;H)} \leq c_j(\kappa; \lambda_0) \left\| \hat{g}(\xi) \right\|_{L_2(R;H)}, j = 1, 2, 3. \tag{16}$$

And inequalities (16), in turn, are equivalent to inequalities

$$\left\| A^j \left(\frac{d}{dt} + \frac{\kappa}{2} \right)^{3-j} v \right\|_{L_2(R;H)} \leq c_j(\kappa; \lambda_0) \left\| \mathrm{P}_{0,\kappa} v \right\|_{L_2(R;H)}, j = 1, 2, 3.$$

This proves the theorem. ◄

Theorem 3 has the following direct corollary.

Corollary 3. *The numbers* $N_j \leq c_j(\kappa; \lambda_0)$, $j = 1, 2, 3$.

4. Conditions for solvability of equation (1) in space $W_{2,\kappa}^3(R; H)$

The above results allow us to formulate the exact theorem on the regular solvability of equation (1) with the help of the properties of its operator coefficients.

Theorem 4. *Let* $A = A^* \geq \lambda_0 E$, $\lambda_0 > 0$, $\kappa < 2\lambda_0$ *and* $A_j A^{-j} \in L(H)$, $j = 1, 2, 3$, *and the inequality*

$$c_1(\kappa; \lambda_0) \left\| A_1 A^{-1} \right\|_{H \to H} + c_2(\kappa; \lambda_0) \left\| A_2 A^{-2} \right\|_{H \to H} + c_3(\kappa; \lambda_0) \left\| A_3 A^{-3} \right\|_{H \to H} < 1,$$

hold, where $c_j(\kappa; \lambda_0)$, $j = 1, 2, 3$, *are determined in Theorem 3. Then equation* (1) *is regularly solvable.*

5. Appendix

We illustrate the obtained solvability conditions with an example of the problem for partial differential equations.

Consider the following problem on the strip $R \times [0, \pi]$:

$$-\left(\frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2} \right)^3 u(t, x) + p(x) \frac{\partial^4 u(t, x)}{\partial x^2 \partial t^2} + q(x) \frac{\partial^5 u(t, x)}{\partial x^4 \partial t} +$$

$$+ r(x) \frac{\partial^6 u(t, x)}{\partial x^6} = f(t, x), \tag{17}$$

$$\frac{\partial^{2s} u(t, 0)}{\partial x^{2s}} = \frac{\partial^{2s} u(t, \pi)}{\partial x^{2s}} = 0, s = 0, 1, 2, \tag{18}$$

where $f(t,x) \in L_{2,\kappa}(R; L_2[0,\pi])$, and $p(x)$, $q(x)$, $r(x)$ are functions bounded on the interval $[0,\pi]$.

It is easy to see that the problem (17), (18) reduces to equation (1) with $H = L_2[0,\pi]$, $A_1 = p(x)\frac{\partial^2}{\partial x^2}$, $A_2 = q(x)\frac{\partial^4}{\partial x^4}$, $A_3 = r(x)\frac{\partial^6}{\partial x^6}$, and the operator A is defined on $L_2[0,\pi]$ by the equality $Au = -\frac{d^2u}{dx^2}$ with conditions $u(0) = u(\pi) = 0$. In this case $\lambda_0 = 1$.

Applying Theorem 4, we obtain that if $\kappa < 2$, then under the condition

$$c_1(\kappa;1)\sup_{x\in[0,\pi]}|p(x)| + c_2(\kappa;1)\sup_{x\in[0,\pi]}|q(x)| + c_3(\kappa;1)\sup_{x\in[0,\pi]}|r(x)| < 1,$$

where

$$c_1(\kappa;1) = \begin{cases} \frac{2}{3^{3/2}(1-\kappa)^{1/2}}, & \kappa < 1, \\ \frac{2\kappa^2}{(2-\kappa)^3}, & 1 \le \kappa < 2, \end{cases} \quad c_2(\kappa;1) = \begin{cases} \frac{2}{3^{3/2}(1-\kappa)}, & \kappa < 1, \\ \frac{4|\kappa|}{(2-\kappa)^3}, & 1 \le \kappa < 2, \end{cases}$$

$$c_3(\kappa;1) = \left(1 - \frac{\kappa}{2}\right)^{-3},$$

the problem (17), (18) has a unique solution from the space $W^{3,6}_{t,x,2,\kappa}(R; L_2[0,\pi])$.

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