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Noncommutative Chebyshev Inequality Involving the Hadamard Product

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Abstract. We present several operator extensions of the Chebyshev inequality for Hilbert space operators. The main version deals with the synchronous Hadamard property for Hilbert space operators. Among other inequalities, it is shown that if $\mathfrak{A}$ is a C^*-algebra, T is a compact Hausdorff space equipped with a Radon measure μ as a totally ordered set, then

$$\int_T \alpha(s)d\mu(s)\int_T \alpha(t)(A_t \circ B_t)d\mu(t) \geq \Big(\int_T \alpha(t)(A_t m_{r,\alpha} B_t)d\mu(t)\Big)\circ\Big(\int_T \alpha(s)(A_s m_{r,1-\alpha} B_s)d\mu(s)\Big),$$

where $\alpha \in [0,1]$, $r \in [-1,1]$ and $(A_t)_{t\in T}$, $(B_t)_{t\in T}$ are positive increasing fields in $\mathcal{C}(T,\mathfrak{A})$.

Key Words and Phrases: Chebyshev inequality, Hadamard product, Bochner integral, operator mean.

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1. Introduction and preliminaries

Let $\mathbb{B}(\mathfrak{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathfrak{H}$. In the case where $\dim\mathfrak{H} = n$, we identify $\mathbb{B}(\mathfrak{H})$ with the matrix algebra $\mathbb{M}_n$ of all $n \times n$ matrices with entries in the complex field. An operator $A \in \mathbb{B}(\mathfrak{H})$ is called positive ($A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathfrak{H}$. The set of all positive operators is denoted by $\mathbb{B}(\mathfrak{H})_+$. For selfadjoint operators $A, B \in \mathbb{B}(\mathfrak{H})$, we say $B \geq A$ if $B - A \geq 0$.

The Gelfand map $f(t) \mapsto f(A)$ is an isometric $*$-isomorphism between the C^*-algebra $C(\mathrm{sp}(A))$ of continuous functions on the spectrum $\mathrm{sp}(A)$ of a selfadjoint operator A and the C^*-algebra generated by A and the identity operator I. If $f, g \in C(\mathrm{sp}(A))$, then $f(t) \geq g(t)$ $(t \in \mathrm{sp}(A))$ implies that $f(A) \geq g(A)$.

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Let f be a continuous real valued function on an interval J. The function f is called operator monotone if $A \le B$ implies $f(A) \le f(B)$ for all $A, B \in \mathbb{B}(\mathfrak{H})$ with spectra in J. Given an orthonormal basis $\{e_j\}$ of a Hilbert space $\mathfrak{H}$, the Hadamard product $A \circ B$ of two operators $A, B \in \mathbb{B}(\mathfrak{H})$ is defined by $\langle A \circ Be_i, e_j\rangle = \langle Ae_i, e_j\rangle\langle Be_i, e_j\rangle$. It is known that the Hadamard product can be presented by filtering the tensor product $A \otimes B$ through a positive linear map. In fact, $A \circ B = U^*(A \otimes B)U$, where $U : \mathfrak{H} \to \mathfrak{H} \otimes \mathfrak{H}$ is the isometry defined by $Ue_j = e_j \otimes e_j$; see [1, 2, 9]. For matrices, one easily observe [14] that the Hadamard product of $A = (a_{ij})$ and $B = (b_{ij})$ is $A \circ B = (a_{ij}b_{ij})$, a principal submatrix of the tensor product $A \otimes B = (a_{ij}B)_{1\le i,j\le n}$. From now on when we deal with the Hadamard product of operators, we explicitly assume that we fix an orthonormal basis.

The axiomatic theory of operator means has been developed by Kubo and Ando [10]. An operator mean is a binary operation σ defined on the set of strictly positive operators, if the following conditions hold:

(1) $A \le C, B \le D$ imply $A\sigma B \le C\sigma D$;

(2) $A_n \downarrow A, B_n \downarrow B$ imply $A_n\sigma B_n \downarrow A\sigma B$, where $A_n \downarrow A$ means that $A_1 \ge A_2 \ge \cdots$ and $A_n \to A$ as $n \to \infty$ in the strong operator topology;

(3) $T^*(A\sigma B)T \le (T^*AT)\sigma(T^*BT) \quad (T \in \mathbb{B}(\mathfrak{H}))$;

(4) $I\sigma I = I$.

There exists an affine order isomorphism between the class of operator means and the class of positive operator monotone functions f defined on $(0,\infty)$ with $f(1) = 1$ via $f(t)I = I\sigma(tI)$ $(t > 0)$. In addition, $A\sigma B = A^{\frac{1}{2}} f(A^{\frac{-1}{2}} BA^{\frac{-1}{2}})A^{\frac{1}{2}}$ for all strictly positive operators A, B. The operator monotone function f is called the representing function of σ. Using a limit argument by $A_\varepsilon = A + \varepsilon I$, one can extend the definition of $A\sigma B$ to positive operators. An operator mean σ is symmetric if $A\sigma B = B\sigma A$ for all $A, B \in \mathbb{B}(\mathfrak{H})_+$. For a symmetric operator mean σ, a parametrized operator mean σ_t, $0 \le t \le 1$ is called an interpolational path for σ if it satisfies

(1) $A\sigma_0 B = A$, $A\sigma_{1/2}B = A\sigma B$, and $A\sigma_1 B = B$;

(2) $(A\sigma_p B)\sigma(A\sigma_q B) = A\sigma_{\frac{p+q}{2}} B$ for all $p, q \in [0, 1]$;

(3) The map $t \in [0, 1] \mapsto A\sigma_t B$ is norm continuous for each A and B.

It is easy to see that the set of all $r \in [0, 1]$ satisfying

$$(A\sigma_p B)\sigma_r(A\sigma_q B) = A\sigma_{rp+(1-r)q}B \tag{1}$$

for all p, q is a convex subset of $[0, 1]$ including 0 and 1. The power means interpolational paths are

$$Am_{r,t}B = A^{\frac{1}{2}}\left(1 - t + t(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^r\right)^{\frac{1}{r}}A^{\frac{1}{2}} \qquad (t \in [0, 1]).$$

In particular, we have the operator weighted arithmetic mean $Am_{1,t}B = A\nabla_t B = (1-t)A+tB$, the operator weighted geometric mean $Am_{0,t}B = A\sharp_t B$ and the operator weighted harmonic mean $Am_{-1,t}B = A!_tB = \left((1-t)A^{-1} + tB^{-1}\right)^{-1}$. The representing function $F_{r,t}$ for $m_{r,t}$ is defined as $F_{r,t}(x) = 1m_{r,t}x = (1 - t + tx^r)^{\frac{1}{r}}$ $(x > 0)$.; see [1, 2] and references therein. Let us consider the real sequences $a = (a_1, \cdots, a_n)$, $b = (b_1, \cdots, b_n)$ and the non-negative sequence $w = (w_1, \cdots, w_n)$. Then the weighed Chebyshev function is defined by $T(w; a, b) := \sum_{i=1}^n w_i \sum_{i=1}^n w_j a_j b_j - \sum_{i=1}^n w_i a_i \sum_{j=1}^n w_j b_j$. In 1882, Chebyshev [6] proved that if a and b are monotone in the same sense, then

$$\sum_{i=1}^n \omega_i \sum_{j=1}^n \omega_j a_j b_j \geq \sum_{i=1}^n \omega_i a_i \sum_{j=1}^n \omega_j b_j. \tag{2}$$

Behdzed in [5] extended inequality (2) to

$$\sum_{i=1}^n \omega_i \sum_{j=1}^n \nu_j a_j b_j + \sum_{i=1}^n \nu_i \sum_{j=1}^n \omega_j a_j b_j \geq \sum_{i=1}^n \nu_i a_i \sum_{j=1}^n \omega_j b_j + \sum_{i=1}^n \omega_i a_i \sum_{j=1}^n \nu_j b_j, \tag{3}$$

where $a_1 \leq \cdots \leq a_n$, $b_1 \leq \cdots \leq b_n$ and $\omega_1, \cdots, \omega_n$, $\nu_1, \cdots, \nu_n$ are nonnegative real numbers. Some integral generalizations of the Chebyshev inequality were given by Barza, Persson and Soria [4]. The Chebyshev inequality is a complement of the Grüss inequality; see [11] and references therein. Dragomir presented some Chebyshev inequalities for selfadjoint operators acting on Hilbert spaces in [8, 7].

A related notion is synchronicity. Recall that two continuous functions $f, g : J \to \mathbb{R}$ are synchronous on an interval J, if

$$\big(f(t) - f(s)\big)\big(g(t) - g(s)\big) \geq 0$$

for all $s, t \in J$. It is obvious that if f, g are monotonic and have the same monotonicity, then they are synchronic.

Let $\mathfrak{A}$ be a C^*-algebra of operators acting on a Hilbert space, T be a compact Hausdorff space and $\mu(t)$ be a Radon measure on T. A field $(A_t)_{t\in T}$ of operators in $\mathfrak{A}$ is called a continuous field of operators if the function $t \mapsto A_t$ is norm continuous on T and the function $t \mapsto \|A_t\|$ is integrable. One can form the Bochner integral $\int_T A_t \mathrm{d}\mu(t)$, which is the unique element in $\mathfrak{A}$ such that

$\varphi\left(\int_T A_t \mathrm{d}\mu(t)\right) = \int_T \varphi(A_t)\mathrm{d}\mu(t)$ for every linear functional φ in the norm dual $\mathfrak{A}^*$ of $\mathfrak{A}$. By [12] for operators $B, A_t \in \mathfrak{A}$ we have

$$\int_T (A_t \circ B) d\mu(t) = \int_T A_t d\mu(t) \circ B \qquad (A_t, B \in \mathfrak{A}). \tag{4}$$

We say that two fields $(A_t)_{t\in T}$ and $(B_t)_{t\in T}$ have the synchronous Hadamard property if

$$\big(A_t - A_s\big) \circ \big(B_t - B_s\big) \geq 0$$

for all $s, t \in T$. We say (A_t) is an increasing (decreasing, resp.) field, whenever $t \preceq s$ implies that $A_t \leq A_s$ ($A_t \geq A_s$, resp.).

In [12], the authors showed that

$$\int_T \alpha(s)d\mu(s) \int_T \alpha(t)(A_t \circ B_t)d\mu(t) \geq \Big(\int_T \alpha(t)A_t d\mu(t)\Big) \circ \Big(\int_T \alpha(s)B_s d\mu(s)\Big), \tag{5}$$

where $\mathfrak{A}$ is a C^*-algebra, T is a compact Hausdorff space equipped with a Radon measure μ, $(A_t)_{t\in T}$ and $(B_t)_{t\in T}$ are fields in $\mathcal{C}(T, \mathfrak{A})$ with the synchronous Hadamard property and $\alpha : T \to [0, +\infty)$ is a measurable function. They also presented

$$\int_T \alpha(s)d\mu(s) \int_T \alpha(t)(A_t \circ B_t)d\mu(t) \geq$$

$$\Big(\int_T \alpha(t)(A_t \sharp_\mu B_t)d\mu(t)\Big) \circ \Big(\int_T \alpha(s)(A_s \sharp_{1-\mu} B_s)d\mu(s)\Big), \tag{6}$$

where $\mathfrak{A}$ is a C^*-algebra, T is a compact Hausdorff space equipped with a Radon measure μ as a totally ordered set, $(A_t)_{t\in T}, (B_t)_{t\in T}$ are positive increasing fields in $\mathcal{C}(T, \mathfrak{A})$, $\alpha : T \to [0, +\infty)$ is a measurable function and $\mu \in [0, 1]$.

In this paper, we provide several operator extensions of the Chebyshev inequality of the form (5) and (6). We present our main results dealing with the Hadamard product for Hilbert space operators.

2. Chebyshev inequality involving Hadamard product

This section is devoted to the presentation of some operator Chebyshev inequalities dealing with the Hadamard product. The first result reads as follows.

Theorem 1. *Let $\mathfrak{A}$ be a C^*-algebra, T be a compact Hausdorff space equipped with a Radon measure μ, let $(A_t)_{t\in T}$ and $(B_t)_{t\in T}$ be fields in $\mathcal{C}(T,\mathfrak{A})$ with the synchronous Hadamard property and let $\alpha,\beta : T\to[0,+\infty)$ be measurable functions. Then*

$$\int_T \alpha(s)d\mu(s)\int_T \beta(t)(A_t\circ B_t)d\mu(t)+\int_T \beta(s)d\mu(s)\int_T \alpha(t)(A_t\circ B_t)d\mu(t)\geq$$

$$\Big(\int_T \alpha(t)A_td\mu(t)\Big)\circ\Big(\int_T \beta(s)B_sd\mu(s)\Big)+\Big(\int_T \beta(t)A_td\mu(t)\Big)\circ\Big(\int_T \alpha(s)B_sd\mu(s)\Big). \tag{7}$$

Proof. We put

$$\Lambda=\int_T \alpha(s)d\mu(s)\int_T \beta(t)(A_t\circ B_t)d\mu(t)+\int_T \beta(s)d\mu(s)\int_T \alpha(t)(A_t\circ B_t)d\mu(t)$$

$$-\Big(\int_T \alpha(t)A_td\mu(t)\Big)\circ\Big(\int_T \beta(s)B_sd\mu(s)\Big)-\Big(\int_T \beta(t)A_td\mu(t)\Big)\circ\Big(\int_T \alpha(s)B_sd\mu(s)\Big).$$

Then

$$\begin{aligned}
\Lambda &= \int_T\int_T \alpha(s)\beta(t)(A_t\circ B_t)d\mu(t)d\mu(s)+\int_T\int_T \beta(s)\alpha(t)(A_t\circ B_t)d\mu(t)d\mu(s)\\
&\quad -\int_T\Big(\int_T \alpha(t)A_td\mu(t)\Big)\circ\beta(s)B_sd\mu(s)\int_T\Big(\int_T \beta(t)A_td\mu(t)\Big)\circ\alpha(s)B_sd\mu(s)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(by (4))}\\
&= \int_T\int_T \alpha(s)\beta(t)(A_t\circ B_t)d\mu(t)d\mu(s)+\int_T\int_T \beta(s)\alpha(t)(A_t\circ B_t)d\mu(t)d\mu(s)\\
&\quad -\int_T\int_T \alpha(t)\beta(s)(A_t\circ B_s)d\mu(t)d\mu(s)-\int_T\int_T \beta(t)\alpha(s)(A_t\circ B_s)d\mu(t)d\mu(s)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(by (4))}\\
&= \int_T\int_T\Big[\alpha(s)\beta(t)(A_t\circ B_t)+\beta(s)\alpha(t)(A_t\circ B_t)\\
&\quad -\alpha(t)\beta(s)(A_t\circ B_s)-\beta(t)\alpha(s)(A_t\circ B_s)\Big]d\mu(t)d\mu(s)\\
&= \frac{1}{2}\Bigg(\int_T\int_T\Big[\alpha(s)\beta(t)(A_t\circ B_t)+\beta(s)\alpha(t)(A_t\circ B_t)\\
&\quad -\alpha(t)\beta(s)(A_t\circ B_s)-\beta(t)\alpha(s)(A_t\circ B_s)\Big]d\mu(t)d\mu(s)\\
&\quad +\int_T\int_T\Big[\alpha(s)\beta(t)(A_t\circ B_t)+\beta(s)\alpha(t)(A_t\circ B_t)
\end{aligned}$$

$$-\alpha(t)\beta(s)(A_t\circ B_s)-\beta(t)\alpha(s)(A_t\circ B_s)\Big]d\mu(t)d\mu(s)\Bigg)$$

Now, if we interchange s with t in the second expression of the last equation, then we get

$$\begin{aligned}
\Lambda \geq\ & \frac{1}{2}\Bigg(\int_T\int_T\Big[\alpha(s)\beta(t)(A_t\circ B_t)+\beta(s)\alpha(t)(A_t\circ B_t)\\
&-\alpha(t)\beta(s)(A_t\circ B_s)-\beta(t)\alpha(s)(A_t\circ B_s)\Big]d\mu(t)d\mu(s)\\
&+\int_T\int_T\Big[\alpha(t)\beta(s)(A_s\circ B_s)+\beta(t)\alpha(s)(A_s\circ B_s)\\
&-\alpha(s)\beta(t)(A_s\circ B_t)-\beta(s)\alpha(t)(A_s\circ B_t)\Big]d\mu(s)d\mu(t)\Bigg)\\
=\ & \frac{1}{2}\Bigg(\int_T\int_T\Big[\alpha(s)\beta(t)(A_t\circ B_t)+\beta(s)\alpha(t)(A_t\circ B_t)\\
&-\alpha(t)\beta(s)(A_t\circ B_s)-\beta(t)\alpha(s)(A_t\circ B_s)\Big]d\mu(t)d\mu(s)\\
&+\int_T\int_T\Big[\alpha(t)\beta(s)(A_s\circ B_s)+\beta(t)\alpha(s)(A_s\circ B_s)\\
&-\alpha(s)\beta(t)(A_s\circ B_t)-\beta(s)\alpha(t)(A_s\circ B_t)\Big]d\mu(t)d\mu(s)\Bigg)\\
=\ & \frac{1}{2}\int_T\int_T\Big[\beta(s)\alpha(t)\big(A_t-A_s\big)\circ\big(B_t-B_s\big)\\
&+\alpha(s)\beta(t)\big(A_t-A_s\big)\circ\big(B_t-B_s\big)\Big]d\mu(t)d\mu(s)
\end{aligned}$$

≥ 0. (since the fields (A_t) and, (B_t) have the synchronous Hadamard property)

◄

In the discrete case $T=\{1,\cdots,n\}$, let $\alpha(i)=\omega_i$ and $\beta(i)=\nu_i$, where $\omega_i,\nu_i\geq 0$ $(1\leq i\leq n)$. Then Theorem 1 yields the following corollary.

Corollary 1. *Suppose that $A_j, B_j\in\mathbb{B}(\mathfrak{H})$ $(1\leq j\leq n)$ are selfadjoint operators with the synchronous Hadamard property and $\omega_1,\cdots,\omega_n$, $\nu_1,\cdots,\nu_n$ are positive numbers. Then*

$$\sum_{i=1}^n\omega_i\sum_{j=1}^n\nu_j(A_j\circ B_j)+\sum_{i=1}^n\nu_i\sum_{j=1}^n\omega_j(A_j\circ B_j)$$

$$\geq \Big(\sum_{i=1}^{n}\omega_i A_i\Big)\circ\Big(\sum_{j=1}^{n}\nu_j B_j\Big)+\Big(\sum_{i=1}^{n}\nu_i A_i\Big)\circ\Big(\sum_{j=1}^{n}\omega_j B_j\Big). \tag{8}$$

Example 1. *If $f_1, f_2, g_1, g_2 \in \mathbf{L}^1(\mathbb{R})$ so that f_1, f_2 are increasing and g_1, g_2 are decreasing on $\mathbb{R}$, then we put $A_t = \begin{pmatrix} f_1(t) & h(t) \\ 0 & g_1(t) \end{pmatrix}$ and $B_t = \begin{pmatrix} f_2(t) & 0 \\ k(t) & g_2(t) \end{pmatrix}$ where $h, k \in \mathbf{L}^1(\mathbb{R})$ are arbitrary and $t \in \mathbb{R}$. From $\big(f_1(t) - f_1(s)\big)\big(f_2(t) - f_2(s)\big)$ and $\big(g_1(t) - g_1(s)\big)\big(g_2(t) - g_2(s)\big)$ being positive for all $s, t \in \mathbb{R}$ it follows that the matrix*

$$\big(A_t - A_s\big)\circ\big(B_t - B_s\big) =$$

$$= \begin{pmatrix} (f_1(t) - f_1(s))\,(f_2(t) - f_2(s)) & 0 \\ 0 & (g_1(t) - g_1(s))\,(g_2(t) - g_2(s)) \end{pmatrix}$$

is positive. Using Theorem 1 we have the inequality

$$\begin{aligned}
&\begin{pmatrix} \int_{\mathbb{R}}\alpha(s)ds\int_{\mathbb{R}}\beta(t)f_1(t)f_2(t)dt & 0 \\ 0 & \int_{\mathbb{R}}\alpha(s)ds\int_{\mathbb{R}}\beta(t)g_1(t)g_2(t)dt \end{pmatrix}\\
&\quad+\begin{pmatrix} \int_{\mathbb{R}}\beta(s)ds\int_{\mathbb{R}}\alpha(t)f_1(t)f_2(t)dt & 0 \\ 0 & \int_{\mathbb{R}}\beta(s)ds\int_{\mathbb{R}}\alpha(t)g_1(t)g_2(t)dt \end{pmatrix}\\
&\geq \begin{pmatrix} \int_{\mathbb{R}}\int_{\mathbb{R}}\alpha(t)\beta(s)f_1(t)f_2(s)dtds & 0 \\ 0 & \int_{\mathbb{R}}\alpha(t)\beta(s)g_1(t)g_2(s)dtds \end{pmatrix}\\
&\quad+\begin{pmatrix} \int_{\mathbb{R}}\int_{\mathbb{R}}\alpha(s)\beta(t)f_1(t)f_2(s)dtds & 0 \\ 0 & \int_{\mathbb{R}}\alpha(s)\beta(t)g_1(t)g_2(s)dtds \end{pmatrix},
\end{aligned}$$

where $\alpha, \beta \in \mathbf{L}^1(\mathbb{R})$.

Let us consider $A_j, B_j \in \mathbb{B}(\mathfrak{H})$ $(1 \leq j \leq n)$ and the nonnegative numbers $\omega_1, \cdots, \omega_n$, $\nu_1, \cdots, \nu_n$ such that $W_n = \sum_{j=1}^{n}\omega_j$, $V_n = \sum_{j=1}^{n}\nu_j$. We define the mapping $Q : N_+ \times N_+ \to \mathbb{B}(\mathfrak{H})$ as follows:

$$\begin{aligned}
Q(k, n, A_j, B_j) &= W_k\sum_{j=1}^{k}\nu_j(A_j\circ B_j) + V_k\sum_{j=1}^{n}\omega_j(A_j\circ B_j)\\
&\quad+\Big(\sum_{i=k+1}^{n}\omega_i A_i\Big)\circ\Big(\sum_{j=1}^{n}\nu_j B_j\Big)+\Big(\sum_{i=1}^{n}\nu_i A_i\Big)\circ\Big(\sum_{j=k+1}^{n}\omega_j B_j\Big)\\
&\quad+\Big(\sum_{i=k+1}^{n}\nu_i A_i\Big)\circ\Big(\sum_{j=1}^{n}\omega_j B_j\Big)+\Big(\sum_{i=1}^{n}\omega_i A_i\Big)\circ\Big(\sum_{j=k+1}^{n}\omega_j B_j\Big),
\end{aligned}$$

where $k = 1, 2, \cdots, n$, and

$$\sum_{j=n+1}^{n} \omega_j A_j = \sum_{j=n+1}^{n} \omega_j B_j = \sum_{j=n+1}^{n} \nu_j A_j = \sum_{j=n+1}^{n} \nu_j B_j = 0. \tag{9}$$

Using the definition of Q and the relation (9) we get
(a) $Q(1, n, A_j, B_j) = \Big(\sum_{i=1}^{n} \omega_i A_i \Big) \circ \Big(\sum_{j=1}^{n} \nu_j B_j \Big) + \Big(\sum_{i=1}^{n} \nu_i A_i \Big) \circ \Big(\sum_{j=1}^{n} \omega_j B_j \Big)$
(b) $Q(n, n, A_j, B_j) = \sum_{i=1}^{n} \omega_i \sum_{j=1}^{n} \nu_j (A_j \circ B_j) + \sum_{i=1}^{n} \nu_i \sum_{j=1}^{n} \omega_j (A_j \circ B_j)$.

Now, in the next theorem we show a refinement of inequality (8).

Theorem 2. *Suppose that $A_j, B_j \in \mathbb{B}(\mathfrak{H})$ $(1 \leq j \leq n)$ are selfadjoint operators with the synchronous Hadamard property, Q is defined as above, $\omega_1, \cdots, \omega_n$, $\nu_1, \cdots, \nu_n$ are positive numbers. Then*

$$Q(n, n, A_j, B_j) \geq \cdots \geq Q(k, n, A_j, B_j) \geq \cdots \geq Q(1, n, A_j, B_j)$$

for each $k = 1, 2, \cdots, n$.

Proof. For all $k = 2, 3, \cdots, n$, we have

$$Q(k, n, A_j, B_j) - Q(k-1, n, A_j, B_j) =$$

$$= (W_{k-1} + \omega_k) \left(\sum_{j=1}^{k-1} \nu_j (A_j \circ B_j) + \nu_k A_k \circ B_k \right) +$$

$$+ (V_{k-1} + \nu_k) \left(\sum_{j=1}^{k-1} \omega_j (A_j \circ B_j) + \omega_k A_k \circ B_k \right)$$

$$\left[W_{k-1} \sum_{j=1}^{k-1} \nu_j (A_j \circ B_j) + V_{k-1} \sum_{j=1}^{k-1} \omega_j (A_j \circ B_j) \right] + \left(\sum_{j=k+1}^{n} \omega_j A_j \right) \circ \left(\sum_{j=1}^{n} \nu_j B_j \right)$$

$$+ \left(\omega_k A_k + \sum_{j=1}^{k-1} \omega_j A_j \right) \circ \sum_{j=k+1}^{n} \nu_j B_j - \left(\omega_k A_k + \sum_{j=k+1}^{n} \omega_j A_j \right) \circ \sum_{j=1}^{n} \nu_j B_j$$

$$- \sum_{j=1}^{k-1} \omega_j A_j \circ \left(\nu_k B_k + \sum_{j=k+1}^{n} \nu_j B_j \right) + \left(\sum_{j=k+1}^{n} \nu_j A_j \right) \circ \left(\sum_{j=1}^{n} \omega_j B_j \right) +$$

$$\left(\nu_k A_k + \sum_{j=1}^{k-1} \nu_j A_j\right) \circ \sum_{j=k+1}^{n} \omega_j B_j - \left(\nu_k A_k + \sum_{j=k+1}^{n} \nu_j A_j\right) \circ \sum_{j=1}^{n} \omega_j B_j$$

$$- \sum_{j=1}^{k-1} \omega_j B_j \circ \left(\omega_k B_k + \sum_{j=1}^{n} \omega_j B_j\right)$$

$$= \left[\omega_j \sum_{j=1}^{k-1} \nu_j (A_j \circ B_j) + \omega_k (A_k \circ B_k) \sum_{j=1}^{k-1} \nu_j \right.$$

$$\left. - \left(\omega_k A_k \circ \sum_{j=1}^{k-1} \nu_j B_j\right) - \left(\omega_k B_k \circ \sum_{j=1}^{k-1} \nu_j A_j\right)\right]$$

$$+ \left[\nu_k \sum_{j=1}^{k-1} \omega_j (A_j \circ B_j) + \nu_k (A_k \circ B_k) \sum_{j=1}^{k-1} \omega_j \right.$$

$$\left. - \left(\nu_k A_k \circ \sum_{j=1}^{k-1} \omega_j B_j\right) - \left(\nu_k B_k \circ \sum_{j=1}^{k-1} \omega_j A_j\right)\right]$$

$$= \omega_k \sum_{j=1}^{k-1} \nu_j (A_k - A_j) \circ (B_k - B_j) + \nu_k \sum_{j=1}^{k-1} \omega_j (A_k - A_j) \circ (B_k - B_j) \geq 0$$

(since the sequences $(A_j)_{j=1}^n$ and $(B_j)_{j=1}^n$ have synchronous Hadamard property).

◀

Remark 1. *If we put $\omega_j = \nu_j$ $(1 \leq j \leq n)$ in Theorem 2, then we get the inequalities*

$$q(n, n, A_j, B_j) \geq \cdots \geq q(k, n, A_j, B_j) \geq \cdots \geq q(1, n, A_j, B_j),$$

where

$$q(k, n, A_j, B_j) = \sum_{j=1}^{k} \omega_j \sum_{j=1}^{k} \omega_j (A_j \circ B_j) + \Big(\sum_{i=k+1}^{n} \omega_i A_i\Big) \circ \Big(\sum_{j=1}^{n} \omega_j B_j\Big)$$

$$+ \Big(\sum_{i=k+1}^{n} \omega_i A_i\Big) \circ \Big(\sum_{j=1}^{n} \omega_j B_j\Big) \qquad (k = 1, 2, \cdots, n).$$

In the next result, we show an extension of (6) for interpolational means.

Theorem 3. *Let $\mathfrak{A}$ be a C^*-algebra, T be a compact Hausdorff space equipped with a Radon measure μ as a totally ordered set, let $(A_t)_{t\in T}, (B_t)_{t\in T}$ be positive increasing fields in $\mathcal{C}(T,\mathfrak{A})$ and let $\alpha : T \to [0,+\infty)$ be a measurable function. Then*

$$\int_T \alpha(s)d\mu(s)\int_T \alpha(t)(A_t \circ B_t)d\mu(t)$$

$$\geq \Big(\int_T \alpha(t)(A_t m_{r,\alpha} B_t)d\mu(t)\Big) \circ \Big(\int_T \alpha(s)(A_s m_{r,1-\alpha} B_s)d\mu(s)\Big)$$

for all $\alpha \in [0,1]$ and all $r \in [-1,1]$.

Proof. We have

$$\begin{aligned} A_t \circ B_t &= (A_t \circ B_t) m_{r,\alpha} (A_t \circ B_t) = (U^*(A_t \otimes B_t)U) m_{r,\alpha} (U^*(B_t \otimes A_t)U) \\ &\geq U^*((A_t \otimes B_t) m_{r,\alpha} (B_t \otimes A_t))U \geq U^*((A_t m_{r,\alpha} B_t) \otimes (B_t m_{r,\alpha} A_t))U \\ &= (A_t m_{r,\alpha} B_t) \circ (B_t m_{r,\alpha} A_t) = (A_t m_{r,\alpha} B_t) \circ (A_t m_{r,1-\alpha} B_t), \end{aligned} \tag{10}$$

where $t \in T$; see [13, p. 174]. Let $s, t \in T$. Without loss of generality, assume that $s \preceq t$. Then by the property (i) of the operator mean, we have $0 \leq (A_t m_{r,1-\alpha} B_t) - (A_s m_{r,1-\alpha} B_s)$ and $0 \leq (A_t m_{r,\alpha} B_t) - (A_s m_{r,\alpha} B_s)$. Then

$$\begin{aligned} &\int_T \alpha(s)d\mu(s)\int_T \alpha(t)(A_t \circ B_t)d\mu(t) \\ &\qquad - \Big(\int_T \alpha(t)(A_t m_{r,\alpha} B_t)d\mu(t)\Big) \circ \Big(\int_T \alpha(s)(A_s m_{r,1-\alpha} B_s)d\mu(s)\Big) \\ &= \int_T\int_T \alpha(s)\alpha(t)(A_t \circ B_t)d\mu(t)d\mu(s) \\ &\qquad - \int_T\int_T \alpha(t)\alpha(s)\Big((A_t m_{r,\alpha} B_t) \circ (A_s m_{r,1-\alpha} B_s)\Big)d\mu(t)d\mu(s) \text{ (by 4)} \\ &\geq \int_T\int_T \alpha(s)\alpha(t)\Big((A_t m_{r,\alpha} B_t) \circ (A_t m_{r,1-\alpha} B_t)\Big)d\mu(t)d\mu(s) \\ &\quad - \int_T\int_T \alpha(t)\alpha(s)\Big((A_t m_{r,\alpha} B_t) \circ (A_s m_{r,1-\alpha} B_s)\Big)d\mu(t)d\mu(s) \\ &\qquad\qquad\qquad\qquad\qquad \text{(by equation (10))} \end{aligned}$$

$$\begin{aligned} &= \int_T\int_T \alpha(s)\alpha(t)\Big((A_t m_{r,\alpha} B_t) \circ (A_t m_{r,1-\alpha} B_t)\Big) \\ &\quad -\alpha(t)\alpha(s)\Big((A_t m_{r,\alpha} B_t) \circ (A_s m_{r,1-\alpha} B_s)\Big)d\mu(t)d\mu(s) \end{aligned}$$

$$= \frac{1}{2}\Big[\int_T\int_T \alpha(s)\alpha(t)\Big((A_t m_{r,\alpha} B_t) \circ (A_t m_{r,1-\alpha} B_t)\Big)$$
$$-\alpha(t)\alpha(s)\Big((A_t m_{r,\alpha} B_t) \circ (A_s m_{r,1-\alpha} B_s)\Big) d\mu(t) d\mu(s)$$
$$+\int_T\int_T \alpha(t)\alpha(s)\Big((A_s m_{r,\alpha} B_s) \circ (A_s m_{r,1-\alpha} B_s)\Big)$$
$$-\alpha(s)\alpha(t)\Big((A_s m_{r,\alpha} B_s) \circ (A_t m_{r,1-\alpha} B_t)\Big) d\mu(s) d\mu(t)\Big]$$

(interchanging s and t in the second term)

$$= \frac{1}{2}\int_T\int_T \Big[\alpha(s)\alpha(t)\Big((A_t m_{r,t} B_t) \circ (A_t m_{r,1-\alpha} B_t)\Big)$$
$$-\alpha(t)\alpha(s)\Big((A_t m_{r,\alpha} B_t) \circ (A_s m_{r,\alpha} B_s)\Big)$$
$$+\alpha(t)\alpha(s)\Big((A_s m_{r,\alpha} B_s) \circ (A_s m_{r,1-\alpha} B_s)\Big)$$
$$-\alpha(s)\alpha(t)\Big((A_s m_{r,\alpha} B_s) \circ (A_t m_{r,1-\alpha} B_t)\Big)\Big] d\mu(t) d\mu(s)$$

(by equation (4))

$$= \frac{1}{2}\int_T\int_T \alpha(s)\alpha(t)\Big[(A_t m_{r,\alpha} B_t) - (A_s m_{r,\alpha} B_s)\Big]$$
$$\circ\Big[(A_t m_{r,1-\alpha} B_t) - (A_s m_{r,1-\alpha} B_s)\Big] d\mu(t) d\mu(s)$$

(by the property (i) of the operator mean).

◀

In the discrete case $T = \{1, \cdots, n\}$, if $\alpha(i) = \omega_i$ and $\beta(i) = \nu_i$, where $\omega_i, \nu_i \geq 0$ $(1 \leq i \leq n)$, then Theorem 3 yields the following result.

Corollary 2. *Assume that $(A_j)_{j=1}^n, (B_j)_{j=1}^n \in \mathbb{B}(\mathfrak{H})$ are positive increasing sequences and $\omega_1, \cdots, \omega_n$, $\nu_1, \cdots, \nu_n$ are positive numbers. Then*

$$\sum_{i=1}^n w_i \sum_{i=1}^n \nu_i (A_i \circ B_i) \geq \Big(\sum_{i=1}^n w_i (A_i m_{r,\alpha} B_i)\Big) \circ \Big(\sum_{j=1}^n \nu_j (A_j m_{r,1-\alpha} B_j)\Big)$$

for all $\alpha \in [0,1]$ and all $r \in [-1,1]$.

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