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Bézier Curves and Surfaces Based on Modified Bernstein Polynomials

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Abstract. In this paper, Bézier curves and surfaces have been constructed based on modified Bernstein bases functions with shifted knots for $t \in \left[\frac{\alpha}{n+\beta}, \frac{n+\alpha}{n+\beta}\right]$. Various properties of these modified Bernstein bases are studied. A de Casteljau type algorithm has been developed to compute Bézier curves and surfaces with shifted knots. Furthermore, some fundamental properties of Bézier curves and surfaces with modified Bernstein bases are also discussed. Introduction of parameters α and β enable us to shift Bernstein bases functions over subintervals of $[0,1]$. These new curves have some properties similar to classical Bézier curves. We get Bézier curves defined on $[0,1]$ when we set the parameters α, β to the value 0. Simulation study is performed through MATLAB R2010a. It has been concluded that Bézier curves that are generated over any subinterval of $[0,1]$ based on modified Bernstein bases functions are similar to the Bézier curves that are generated based on classical Bernstein bases functions over the interval $[0,1]$.

Key Words and Phrases: degree elevation, degree reduction, de Casteljau algorithm, Bernstein blending functions with shifted knots, Bézier curve, tensor product, shape preserving.

2010 Mathematics Subject Classifications: 65D17, 41A10, 41A25, 41A36

1. Introduction and preliminaries

It was S.N. Bernstein [1] in 1912, who first introduced his famous operators $B_n : C[0,1] \to C[0,1]$ defined for any $n \in \mathbb{N}$ and for any function $f \in C[0,1]$ where $C[0,1]$ denote the set of all continuous functions on $[0,1]$ which is equipped with sup-norm $\|.\|_{C[0,1]}$

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \quad x \in [0,1]. \tag{1}$$

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and named them Bernstein polynomials to prove the Weierstrass theorem [9]. Bernstein showed that if $f \in C[0,1]$, then $B_n(f;x) \rightrightarrows f(x)$ where " $\rightrightarrows$ " represents the uniform convergence. One can find a detailed information about the Bernstein polynomials in [10].

Later it was found that Bernstein polynomials possess many remarkable properties and have various applications in areas such as approximation theory [9], numerical analysis, computer-aided geometric design, and theory of differential equations due to their fine properties of approximation [23].

In computer-aided geometric design (CAGD), Bernstein polynomials and their variants are used in order to preserve the shape of the curves or surfaces. One of the most important curve in CAGD [27] is the classical Bézier curve [2] constructed with the help of Bernstein basis functions. Other works related to different generalizations of Bernstein polynomials and Bézier curves and surfaces can be found in [3, 4, 5, 7, 8, 11, 12, 13, 18, 20, 21, 23, 24, 26]

A new analogue of Bernstein operators using the concept of post quantum calculus ((p,q)-calculus) in approximation theory have been introduced recently by Mursaleen et al. in [12]. Later, based on (p,q)-integers, some approximation results for Bernstein-Stancu operators, Bernstein-Kantorovich operators, (p,q)-Lorentz operators, Bleimann-Butzer and Hahn operators and Bernstein-Shurer operators etc. have also been introduced in [13, 14, 15, 16, 17].

Also see a recent work in approximation theory *On Approximation by Stancu type Jakimovski-Leviatan-Durrmeyer operators* [19].

In [7, 8], Khalid et al. have shown applications of post quantum calculus in computer-aided geometric design in terms of flexibility and applied these Bernstein bases for construction of (p,q)-Bézier curves and surfaces which is further generalization of q-Bézier curves and surfaces [21, 23]. For other relevant works based on Bézier curves, we refer the readers to [4, 5, 11, 18, 21, 22, 23, 25].

In 1968, Stancu [28] showed that the polynomials

$$\big(P_n^{(\alpha,\beta)} f\big)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k+\alpha}{n+\beta}\right), \tag{2}$$

converge to continuous function $f(x)$ uniformly in [0,1] for each real α, β such that $0 \leq \alpha \leq \beta$. The polynomials (2) are called Bernstein-Stancu polynomials.

In 2010, Gadjiev and Ghorbanalizadeh [6] introduced the following construction of Bernstein-Stancu type polynomials with shifted knots:

$$S_{n,\alpha,\beta}(f;x) = \left(\frac{n+\beta_2}{n}\right)^n \sum_{k=0}^{n} \binom{n}{k} \left(x - \frac{\alpha_2}{n+\beta_2}\right)^k \left(\frac{n+\alpha_2}{n+\beta_2} - x\right)^{n-k} f\left(\frac{k+\alpha_1}{n+\beta_1}\right) \tag{3}$$

where $\frac{\alpha_2}{n+\beta_2} \leq x \leq \frac{n+\alpha_2}{n+\beta_2}$ and α_k, β_k $(k = 1, 2)$ are positive real numbers provided $0 \leq \alpha_1 \leq \alpha_2 \leq \beta_1 \leq \beta_2$. It is clear that for $\alpha_2 = \beta_2 = 0$, the polynomials (3) turn into the Bernstein-Stancu polynomials (2) and if $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$, then these polynomials turn into the classical Bernstein polynomials.

In recent years, generalization of the Bézier curve with shape parameters has received continuous attention. Several authors were concerned with the problem of changing the shape of curves and surfaces, while keeping the control polygon unchanged and thus they generalized the Bézier curves in [7, 8, 22, 23].

The outline of this paper are as follows. Section 2 introduces a modified Bernstein functions with shifted knots $G^k_{n,\alpha,\beta}$ and their properties. In Section 3, Bézier curves based on modified Bernstein bases alongwith degree elevation and a de Casteljau algorithm are presented. In Section 4, we define a tensor product patch based on Algorithm 1 and we discuss its geometric properties as well as a degree elevation technique. Furthermore, tensor product of Bézier surfaces on $\left[\frac{\alpha}{n+\beta}, \frac{n+\alpha}{n+\beta}\right] \times \left[\frac{\alpha}{n+\beta}, \frac{n+\alpha}{n+\beta}\right]$ for Bernstein polynomials with shifted knots is introduced and its properties inherited from the univariate case are discussed. In Sections 5 and 6, we have given concluding remarks and MATLAB codes.

In next section, we construct basis functions with shifted knots with the help of (3).

2. Modified Bernstein bases functions with shifted knots

The modified Bernstein bases functions with shifted knots are defined with the help of (3) as follows:

$$G^k_{n,\alpha,\beta}(t) = \binom{n}{k} \frac{1}{\left(\frac{n}{n+\beta}\right)^n} \left(t - \frac{\alpha}{n+\beta}\right)^k \left(\frac{n+\alpha}{n+\beta} - t\right)^{n-k}, \tag{4}$$

where $\frac{\alpha}{n+\beta} \leq t \leq \frac{n+\alpha}{n+\beta}$ and α, β are positive real numbers provided $0 \leq \alpha \leq \beta$.

2.1. Properties

Theorem 1. *The modified Bernstein bases functions with shifted knots possess the following properties*:

1. *Non-negativity*: $G^k_{n,\alpha,\beta}(t) \geq 0 \quad k = 0, 1, \cdots, n, \quad t \in \left[\frac{\alpha}{n+\beta}, \frac{n+\alpha}{n+\beta}\right]$.

2. *Partition of unity*:

$$\sum_{k=0}^{n} G_{n,\alpha,\beta}^{k}\,(t) = 1, \qquad \text{for every} \quad t \in \left[\frac{\alpha}{n+\beta}, \frac{n+\alpha}{n+\beta}\right].$$

3. *End-point interpolation*:

$$G_{n,\alpha,\beta}^{k}\left(\frac{\alpha}{n+\beta}\right) = \begin{cases} 1, & \text{if } k = 0 \\ 0, & \quad k \neq 0 \end{cases}$$

$$G_{n,\alpha,\beta}^{k}\left(\frac{n+\alpha}{n+\beta}\right) = \begin{cases} 1, & \text{if } k = n \\ 0, & \quad k \neq n. \end{cases}$$

4. *Reducibility*: when $\alpha = \beta = 0$, formula (4) reduces to the classical Bernstein basis on $[0, 1]$.

Proof. All these properties can be deduced from (4). ◄

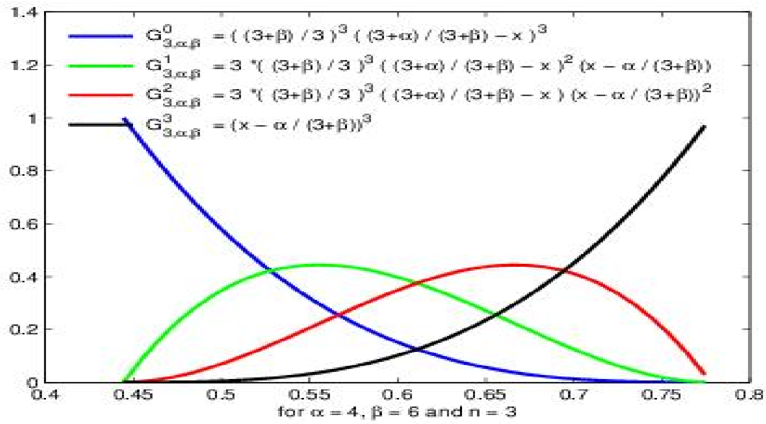


Figure 1: Modified Bernstein bases functions with shifted knots

Figure 1 shows the modified Bernstein bases functions of degree 3 with shifted knots for $\alpha = 4, \beta = 6$. Here we can observe that the sum of blending functions is always unity and also satisfies end point interpolation property. In case $\alpha = \beta = 0$, it turns out to be classical Bernstein basis on $[0, 1]$ which is shown in Figure 2.

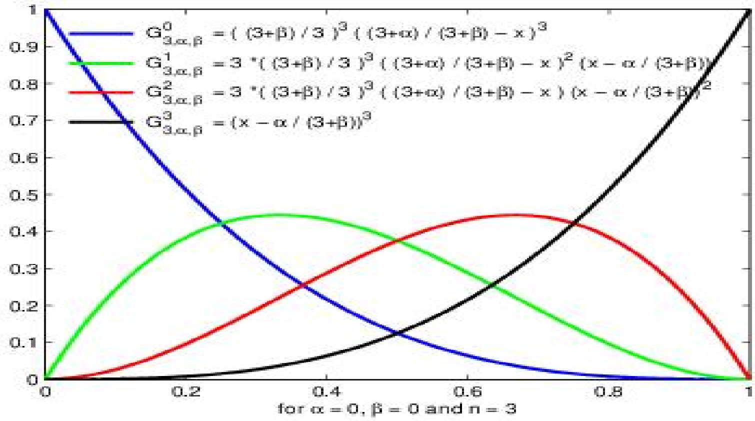


Figure 2: Classical cubic Bernstein bases functions

2.2. Identities

Below we state some important identities which we will use later.

$$\left(\frac{n+\alpha}{n+\beta}-t\right)G^k_{n,\alpha,\beta}\ (t)=\left(\frac{n+1-k}{n+1}\right)\left(\frac{n}{n+\beta}\right)G^k_{n+1,\alpha,\beta}\ (t) \tag{5}$$

and

$$\left(t-\frac{\alpha}{n+\beta}\right)G^k_{n,\alpha,\beta}\ (t)=\left(\frac{n}{n+\beta}\right)\left(\frac{k+1}{n+1}\right)G^{k+1}_{n+1,\alpha,\beta}\ (t). \tag{6}$$

Proof.
Consider

$$\left(\frac{n+\alpha}{n+\beta}-t\right)G^k_{n,\alpha,\beta}(t)=\left(\frac{n+\alpha}{n+\beta}-t\right)\left\{\binom{n}{k}\frac{1}{(\frac{n}{n+\beta})^n}\left(t-\frac{\alpha}{n+\beta}\right)^k\left(\frac{n+\alpha}{n+\beta}-t\right)^{n-k}\right\}$$

$$\left(\frac{n+\alpha}{n+\beta}-t\right)G^k_{n,\alpha,\beta}\ (t)=\frac{\binom{n}{k}}{\binom{n+1}{k}}\binom{n+1}{k}\ \frac{1}{(\frac{n}{n+\beta})^n}\ \left(t-\frac{\alpha}{n+\beta}\right)^k\left(\frac{n+\alpha}{n+\beta}-t\right)^{n-k+1}$$

$$\left(\frac{n+\alpha}{n+\beta}-t\right)G^k_{n,\alpha,\beta}\ (t)=\left\{\frac{\binom{n}{k}}{\binom{n+1}{k}}\ \left(\frac{n}{n+\beta}\right)\right\}G^k_{n+1,\alpha,\beta}\ (t)$$

$$\left(\frac{n+\alpha}{n+\beta}-t\right)G^k_{n,\alpha,\beta}\ (t)=\left(\frac{n+1-k}{n+1}\right)\left(\frac{n}{n+\beta}\right)G^k_{n+1,\alpha,\beta}\ (t).$$

Similarly

$$\left(t-\frac{\alpha}{n+\beta}\right)G^{k}_{n,\alpha,\beta}(t)=\left(\frac{n}{n+\beta}\right)\left(\frac{k+1}{n+1}\right)G^{k+1}_{n+1,\alpha,\beta}(t)$$

$$\left(t-\frac{\alpha}{n+\beta}\right)G^{k}_{n,\alpha,\beta}(t)=\left(x-\frac{\alpha}{n+\beta}\right)\left\{\binom{n}{k}\frac{1}{(\frac{n}{n+\beta})^{n}}\left(t-\frac{\alpha}{n+\beta}\right)^{k}\left(\frac{n+\alpha}{n+\beta}-t\right)^{n-k}\right\}.$$

$$\left(t-\frac{\alpha}{n+\beta}\right)G^{k}_{n,\alpha,\beta}\ (t)=$$

$$\begin{aligned}
&=\left(x-\frac{\alpha}{n+\beta}\right)\left\{\binom{n}{k}\frac{1}{(\frac{n}{n+\beta})^{n}}\left(t-\frac{\alpha}{n+\beta}\right)^{k}\left(\frac{n+\alpha}{n+\beta}-t\right)^{n-k}\right\}\\
&=\binom{n}{k}\frac{1}{(\frac{n}{n+\beta})^{n}}\left(t-\frac{\alpha}{n+\beta}\right)^{k+1}\left(\frac{n+\alpha}{n+\beta}-t\right)^{n-k}\\
&=\frac{\binom{n}{k}}{\binom{n+1}{k+1}}\binom{n+1}{k+1}\frac{1}{(\frac{n}{n+\beta})^{n}}\quad\left(t-\frac{\alpha}{n+\beta}\right)^{k+1}\left(\frac{n+\alpha}{n+\beta}-t\right)^{n-k}\\
&=\left(\frac{n}{n+\beta}\right)\left(\frac{k+1}{n+1}\right)G^{k+1}_{n+1,\alpha,\beta}(t).\ \blacktriangleleft
\end{aligned}$$

Theorem 2. *Every modified Bernstein function with shifted knots of degree n is a linear combination of two modified Bernstein functions with shifted knots of degree $n+1$:*

$$G^{k}_{n,\alpha,\beta}\ (t)=\left(\frac{n+1-k}{n+1}\right)G^{k}_{n+1,\alpha,\beta}\ (t)+\left(\frac{k+1}{n+1}\right)G^{k+1}_{n+1,\alpha,\beta}\ (t)\tag{7}$$

where
$\frac{\alpha}{n+\beta}\leq t\leq\frac{n+\alpha}{n+\beta}$ *and* α,β *are positive real numbers satisfying* $0\leq\alpha\leq\beta$.

Proof.
We have

$$\left(\frac{n}{n+\beta}\right)\ G^{k}_{n,\alpha,\beta}\ (t)=\quad G^{k}_{n,\alpha,\beta}\left(\frac{n+\alpha}{n+\beta}-t+\{t-\frac{\alpha}{n+\beta}\}\right)$$

or

$$\left(\frac{n}{n+\beta}\right)G^{k}_{n,\alpha,\beta}\ (t)=\left(\frac{n+\alpha}{n+\beta}-t\right)G^{k}_{n,\alpha,\beta}\ +\left(t-\frac{\alpha}{n+\beta}\right)G^{k}_{n,\alpha,\beta}.$$

On using equalities (5), (6), we can easily get

$$G^{k}_{n,\alpha,\beta}\,(t)=\left(\frac{n+1-k}{n+1}\right)G^{k}_{n+1,\alpha,\beta}\,(t)+\left(\frac{k+1}{n+1}\right)G^{k+1}_{n+1,\alpha,\beta}\,(t).\ \blacktriangleleft$$

Theorem 3. *Every modified Bernstein function with shifted knots of degree n is a linear combination of two modified Bernstein functions with shifted knots of degree $n-1$:*

$$G^{k}_{n,\alpha,\beta}\,(t)=\frac{n+\beta}{n}\left(t-\frac{\alpha}{n+\beta}\right)G^{k-1}_{n-1,\alpha,\beta}\,(t)+\frac{n+\beta}{n}\left(\frac{n+\alpha}{n+\beta}-t\right)G^{k}_{n-1,\alpha,\beta}\,(t), \quad (8)$$

where
$\frac{\alpha}{n+\beta}\leq t\leq\frac{n+\alpha}{n+\beta}$ *and* α,β *are positive real numbers satisfying* $0\leq\alpha\leq\beta$.

Proof.
On using Pascal type relation $\binom{n}{k}=\binom{n-1}{k-1}+\binom{n-1}{k}$, we get

$$\begin{aligned}
G^{k}_{n,\alpha,\beta}\,(t) &= \binom{n}{k}\ \frac{1}{(\frac{n}{n+\beta})^{n}}\ \left(t-\frac{\alpha}{n+\beta}\right)^{k}\left(\frac{n+\alpha}{n+\beta}-t\right)^{n-k}\\
&=\left\{\binom{n-1}{k-1}+\binom{n-1}{k}\right\}\ \frac{1}{(\frac{n}{n+\beta})^{n}}\ \left(t-\frac{\alpha}{n+\beta}\right)^{k}\left(\frac{n+\alpha}{n+\beta}-t\right)^{n-k}\\
&=\binom{n-1}{k-1}\frac{1}{(\frac{n}{n+\beta})^{n}}\ \left(t-\frac{\alpha}{n+\beta}\right)^{k}\left(\frac{n+\alpha}{n+\beta}-t\right)^{n-k}\\
&+\binom{n-1}{k}\frac{1}{(\frac{n}{n+\beta})^{n}}\ \left(t-\frac{\alpha}{n+\beta}\right)^{k}\left(\frac{n+\alpha}{n+\beta}-x\right)^{n-k}\\
&=\frac{n+\beta}{n}\left(t-\frac{\alpha}{n+\beta}\right)G^{k-1}_{n-1,\alpha,\beta}\,(t)+\frac{n+\beta}{n}\left(\frac{n+\alpha}{n+\beta}-t\right)G^{k}_{n-1,\alpha,\beta}\,(t).
\end{aligned}$$

When $\alpha=\beta=0$, equalities (7), (8) reduce to the degree evaluation formula of the classical Bernstein bases functions. ◀

3. Bézier curves based on modified Bernstein bases

Using modified Bernstein bases functions with shifted knots, the Bézier curves of degree n are defined as follows:

$$\mathbf{P}(t;\alpha,\beta)=\sum_{k=0}^{n}\mathbf{P_k}\ G^{k}_{n,\alpha,\beta}(t), \quad (9)$$

where $\mathbf{P_k} \in \mathbb{R}^3$ $(k = 0, 1, \cdots, n)$, and $\mathbf{P_k}$ are control points. The adjacent points $\mathbf{P_k}$, $k = 0, 1, 2, \cdots, n$ are joined to obtain a polygon which is called the control polygon of Bézier curves.

3.1. Properties

Theorem 4. *Bézier curves based on modified Bernstein bases have the following basic properties*:

1. Bézier curves have geometric and affine invariance.

2. Bézier curves lie inside the convex hull of their control polygons.

3. The end-point interpolation property: $\mathbf{P}(\frac{\alpha}{n+\beta}; \alpha, \beta) = \mathbf{P_0}$, $\mathbf{P}(\frac{n+\alpha}{n+\beta}; \alpha, \beta) = \mathbf{P_n}$.

4. Reducibility: When $\alpha = \beta = 0$ formula (9) reduces to a classical Bézier curve.

Proof. These properties of Bézier curves based on modified Bernstein bases can be easily deduced. ◄

Theorem 5. *The end-point property of derivative*:

$$\mathbf{P}'\left(\frac{\alpha}{n+\beta}; \alpha, \beta\right) = (n+\beta)(\mathbf{P_1} - \mathbf{P_0})\left(\frac{n-1+\beta}{n-1}\right)^{n-1}\left(\frac{n-1+\alpha}{n-1+\beta} - \frac{\alpha}{n+\beta}\right)^{n-1-k} \tag{10}$$

$$\mathbf{P}'\left(\frac{n+\alpha}{n+\beta}; \alpha, \beta\right) = (n+\beta)(\mathbf{P_n} - \mathbf{P_{n-1}})\left(\frac{n-1+\beta}{n-1}\right)^{n-1}\left(\frac{n+\alpha}{n+\beta} - \frac{\alpha}{n-1+\beta}\right)^{n-1} \tag{11}$$

i.e. Bézier curves with shifted knots are tangent to fore-and-aft edges of their control polygons at end points.

Proof.
Let

$$\begin{aligned}
\mathbf{P}(t; \alpha, \beta) &= \sum_{k=0}^{n} \mathbf{P_k}\ G^k_{n,\alpha,\beta}(t) \\
&= \sum_{k=0}^{n} \mathbf{P_k} \binom{n}{k} \ \frac{1}{(\frac{n}{n+\beta})^n} \ \left(t - \frac{\alpha}{n+\beta}\right)^k \left(\frac{n+\alpha}{n+\beta} - t\right)^{n-k}
\end{aligned}$$

$$= \mathbf{V}(t;\alpha,\beta)$$

or

$$\mathbf{P}(t;\alpha,\beta) \ = \mathbf{V}(t;\alpha,\beta).$$

On differentiating both sides with respect to 't', we have

$$\mathbf{P}'(t;\alpha,\beta) \ = \mathbf{V}'(t;\alpha,\beta).$$

Let

$$A_k^n(t;\alpha,\beta) = \binom{n}{k} \frac{1}{(\frac{n}{n+\beta})^n} \left(t - \frac{\alpha}{n+\beta}\right)^k \left(\frac{n+\alpha}{n+\beta} - t\right)^{n-k}.$$

Then

$$\mathbf{V}(t;\alpha,\beta) = \sum_{k=0}^{n} \mathbf{P_k} A_k^n(t;\alpha,\beta).$$

$$\begin{aligned}
(A_k^n(t;\alpha,\beta)' &= \binom{n}{k} \frac{1}{(\frac{n}{n+\beta})^n} \ k \left(t - \frac{\alpha}{n+\beta}\right)^{k-1} \left(\frac{n+\alpha}{n+\beta} - t\right)^{n-k} \\
&- \binom{n}{k} \frac{1}{(\frac{n}{n+\beta})^n} \left(t - \frac{\alpha}{n+\beta}\right)^{k} (n-k) \left(\frac{n+\alpha}{n+\beta} - t\right)^{n-k-1} \\
&= (n+\beta)\{A_{k-1}^{n-1}(t;\alpha,\beta) - A_k^{n-1}(t;\alpha,\beta)\},
\end{aligned}$$

which implies

$$\mathbf{V}'(t;\alpha,\beta) = \sum_{k=0}^{n} \mathbf{P_k}(A_k^n(t;\alpha,\beta)'.$$

Now

$$\mathbf{V}'\left(\frac{\alpha}{n+\beta};\alpha,\beta\right) = \mathbf{P}'(\frac{\alpha}{n+\beta};\alpha,\beta) = (n+\beta)(\mathbf{P_1} - \mathbf{P_0})A_0^{n-1}(t;\alpha,\beta)$$

and

$$\mathbf{P}'\left(\frac{\alpha}{n+\beta};\alpha,\beta\right) = (n+\beta)(\mathbf{P_1} - \mathbf{P_0})\left(\frac{n-1+\beta}{n-1}\right)^{n-1} \left(\frac{n-1+\alpha}{n-1+\beta} - \frac{\alpha}{n+\beta}\right)^{n-1-k}.$$

Similarly after some computation, we have

$$\mathbf{V}'\left(\frac{n+\alpha}{n+\beta};\alpha,\beta\right)=\mathbf{P}'\left(\frac{n+\alpha}{n+\beta};\alpha,\beta\right)=(n+\beta)(\mathbf{P_n}-\mathbf{P_{n-1}})A_{k-1}^{n-1}\left(\frac{n+\alpha}{n+\beta}\right),$$

$$\mathbf{P}'\left(\frac{n+\alpha}{n+\beta};\alpha,\beta\right)=(n+\beta)(\mathbf{P_n}-\mathbf{P_{n-1}})\left(\frac{n-1+\beta}{n-1}\right)^{n-1}\left(\frac{n+\alpha}{n+\beta}-\frac{\alpha}{n-1+\beta}\right)^{n-1}.$$

◀

3.2. Degree elevation for Bézier curves with shifted knots

Bézier curves with shifted knots have a degree elevation algorithm that is similar to that possessed by the classical Bézier curves. Using the technique of degree elevation, we can increase the flexibility of a given curve.

As we know, Bézier curves based on modified Bernstein bases with shifted knots are given by

$$\mathbf{P}(t;\alpha,\beta)=\sum_{k=0}^{n}\mathbf{P_k}\ G_{n,\alpha,\beta}^{k}\ (t).$$

Then after applying degree elevation algorithm on this, we have

$$\mathbf{P}(t;\alpha,\beta)=\sum_{k=0}^{n+1}\mathbf{P_k^*}\ G_{n+1,\alpha,\beta}^{k}\ (t)$$

as Bézier curves based on modified Bernstein bases of degree $n+1$, where

$$\mathbf{P_k^*}=\left(1-\frac{k}{n+1}\right)\mathbf{P_{k-1}}+\left(\frac{k}{n+1}\right)\mathbf{P_k}. \tag{12}$$

The equation above can be derived from Theorem 2. If we denote by $P=(P_0,P_1,\cdots,P_n)^T$ the vector of control points of the initial Bézier curve of degree n, and by $\mathbf{P^{(1)}}=(P_0^*,P_1^*,\cdots,P_{n+1}^*)$ the vector of control points of the degree elevated Bézier curve of degree $n+1$, then we can represent the degree elevation procedure as

$$\mathbf{P^{(1)}}=T_{n+1}\mathbf{P},$$

where

$$T_{n+1} = \frac{1}{n+1}\begin{bmatrix} n+1 & 0 & \dots & 0 & 0 \\ n+1-n & n & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & n+1-2 & 2 & 0 \\ 0 & 0 & \dots & n+1-1 & 1 \\ 0 & 0 & \dots & 0 & n+1 \end{bmatrix}_{(n+2)\times(n+1)}.$$

For any $l \in \mathbb{N}$, the vector of control points of the degree elevated Bézier curve of degree $n+l$ is $\mathbf{P^{(l)}} = T_{n+l}\ T_{n+2}\cdots .T_{n+1}\mathbf{P}$. As $l \longrightarrow \infty$, the control polygon $\mathbf{P^{(l)}}$ converges to a Bézier curve.

3.3. de Casteljau type algorithm for Bézier curves with shifted knots

The recursive de Casteljau type algorithm for Bézier curves with shifted knots of degree n can be expressed as follows:

$$\begin{cases} \mathbf{P_i^0}(t;\alpha,\beta) \equiv \mathbf{P_i^0} \equiv \mathbf{P_i} \quad i = 0,1,2\cdots,n \\ \mathbf{P_i^r}(t;\alpha,\beta) = \frac{n+\beta}{n}\left(t - \frac{\alpha}{n+\beta}\right)\mathbf{P_{i+1}^{r-1}}(t;\alpha,\beta) + \frac{n+\beta}{n}\left(\frac{n+\alpha}{n+\beta} - t\right)\mathbf{P_i^{r-1}}(t;\alpha,\beta) \\ r = 1,\cdots,n, \quad i = 0,1,2\cdots,n-r., \quad \frac{\alpha}{n+\beta} \leq t \leq \frac{n+\alpha}{n+\beta}, \quad 0 \leq \alpha \leq \beta. \end{cases} \tag{13}$$

Then

$$\mathbf{P}(t;\alpha,\beta) = \sum_{i=0}^{n-1}\mathbf{P_i^1}(t;\alpha,\beta) = \cdots = \sum \mathbf{P_i^r}(t;\alpha,\beta)\, G^i_{n-r,\alpha,\beta}(t) = \cdots = \mathbf{P_0^n}\,(t;\alpha,\beta). \tag{14}$$

It is clear that this result can be obtained from Theorem (3). When $\alpha = \beta = 0$, the formulas (13) and (14) recover the de Casteljau algorithms for classical Bézier curves. Let $P^0 = (P_0, P_1, \cdots, P_n)^T$, $P^r = (P_0^r, P_1^r, \cdots, P_{n-r}^r)^T$. Then de Casteljau algorithm can be expressed in matrix form as follows:

$$\mathbf{P^r}(t;\alpha,\beta) = M_r(t;\alpha,\beta)\cdots M_2(t;\alpha,\beta)M_1(t;\alpha,\beta)\mathbf{P^0}, \tag{15}$$

where $M_r(t;\alpha,\beta)$ is a $(n-r+1)\times(n-r+2)$ matrix and

$$M_r(t;\alpha,\beta) =$$

$$= \frac{n+\beta}{n} \begin{bmatrix} \left(\frac{n+\alpha}{n+\beta}-t\right) & \left(t-\frac{\alpha}{n+\beta}\right) & \cdots & 0 & 0 \\ 0 & \left(\frac{n+\alpha}{n+\beta}-t\right) & \left(t-\frac{\alpha}{n+\beta}\right) & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \left(\frac{n+\alpha}{n+\beta}-t\right) & \left(t-\frac{\alpha}{n+\beta}\right) & 0 \\ 0 & 0 & \cdots & \left(\frac{n+\alpha}{n+\beta}-t\right) & \left(t-\frac{\alpha}{n+\beta}\right) \end{bmatrix}.$$

4. Tensor product Bézier surfaces with shifted knots on $\left[\frac{\alpha}{n+\beta}, \frac{n+\alpha}{n+\beta}\right] \times \left[\frac{\alpha}{n+\beta}, \frac{n+\alpha}{n+\beta}\right]$

We define a two-parameter family $\mathbf{P}(u,v)$ of tensor product Bézier surfaces of degree $m \times n$ as follows:

$$\mathbf{P}(u,v) = \sum_{i=0}^{m}\sum_{j=0}^{n} \mathbf{P}_{i,j} G^{i}_{m,\alpha,\beta}(u) G^{j}_{n,\alpha,\beta}(v), (u,v) \in \left[\frac{\alpha}{n+\beta}, \frac{n+\alpha}{n+\beta}\right] \times \left[\frac{\alpha}{n+\beta}, \frac{n+\alpha}{n+\beta}\right], \quad (16)$$

where $\mathbf{P}_{i,j} \in \mathbb{R}^3 \, (i = 0,1,\cdots,m, j = 0,1,\cdots,n)$, and $G^{i}_{m,\alpha,\beta}(u), G^{j}_{n,\alpha,\beta}(v)$ are modified Bernstein functions, respectively. We refer to the $\mathbf{P}_{i,j}$ as the control points. By joining up adjacent points in the same row or column, we obtain a net which is called the control net of tensor product Bézier surface.

4.1. Properties

Bézier surfaces with shifted knots have the following properties:

1. *Geometric invariance and affine invariance property:* Since

$$\sum_{i=0}^{m}\sum_{j=0}^{n} G^{i}_{m,\alpha,\beta}(u) \;\; G^{j}_{n,\alpha,\beta}(v) = 1, \quad (17)$$

   $\mathbf{P}(u,v)$ is an affine combination of its control points.

2. *Convex hull property:* $\mathbf{P}(u,v)$ is a convex combination of $\mathbf{P}_{i,j}$ and lies in the convex hull of its control net.

3. *Isoparametric curves property:* The isoparametric curves $v = v^*$ and $u = u^*$ of a tensor product Bézier surface are the Bézier curves with shifted knots of degree m and degree n, respectively. Namely

$$\mathbf{P}(u,v^*)=\sum_{i=0}^{m}\left(\sum_{j=0}^{n}\mathbf{P}_{i,j}\ G^j_{n,\alpha,\beta}(v^*)\right)G^i_{m,\alpha,\beta}(u)\quad,\quad u\in[\frac{\alpha}{n+\beta},\frac{n+\alpha}{n+\beta}],$$

$$\mathbf{P}(u^*,v)=\sum_{j=0}^{n}\left(\sum_{i=0}^{m}\mathbf{P}_{i,j}\ G^j_{n,\alpha,\beta}(u^*)\right)G^i_{m,\alpha,\beta}(v)\quad,\quad v\in[\frac{\alpha}{n+\beta},\frac{n+\alpha}{n+\beta}].$$

The boundary curves of $\mathbf{P}(u,v)$ are evaluated by $\mathbf{P}(u,\frac{\alpha}{n+\alpha})$, $\mathbf{P}(u,\frac{n+\alpha}{n+\beta})$, $\mathbf{P}(\frac{\alpha}{n+\alpha},v)$ and $\mathbf{P}(\frac{n+\alpha}{n+\beta},v)$.

4. *Corner point interpolation property:* The corner control net coincides with the four corners of the surface. Namely, $\mathbf{P}(\frac{\alpha}{n+\alpha},\frac{\alpha}{n+\alpha})=\mathbf{P}_{0,0}$, $\mathbf{P}(\frac{\alpha}{n+\alpha},\frac{n+\alpha}{n+\beta})=\mathbf{P}_{0,n}$, $\mathbf{P}(\frac{m+\alpha}{m+\beta},\frac{\alpha}{n+\alpha})=\mathbf{P}_{m,0}$, $\mathbf{P}(\frac{m+\alpha}{m+\beta},\frac{n+\alpha}{n+\beta})=\mathbf{P}_{m,n}$.
5. *Reducibility:* When $\alpha=\beta=0$, the formula (16) reduces to a classical tensor product Bézier patch.

4.2. Degree elevation and de Casteljau algorithm

Let $\mathbf{P}(u,v)$ be a tensor product Bézier surface with shifted knots of degree $m\times n$. As an example, let us consider obtaining the same surface as a surface of degree $(m+1)\times(n+1)$. Hence, we need to find new control points $\mathbf{P}^*_{i,j}$ such that

$$\mathbf{P}(u,v)=\sum_{i=0}^{m}\sum_{j=0}^{n}\mathbf{P}_{i,j}G^i_{m,\alpha,\beta}(u)G^j_{n,\alpha,\beta}(v)=\sum_{i=0}^{m+1}\sum_{j=0}^{n+1}\mathbf{P^*}_{i,j}G^i_{m+1,\alpha,\beta}(u)G^j_{n+1,\alpha,\beta}(v).\tag{18}$$

Let $\alpha_i=1-\frac{m+1-i}{m+1}$, $\beta_j=1-\frac{n+1-j}{n+1}$.

Then

$$\mathbf{P}^*_{i,j}=\alpha_i\ \beta_j\ \mathbf{P}_{i-1,j-1}+\alpha_i\ (1-\beta_j)\ \mathbf{P}_{i-1,j}+(1-\alpha_i)\ (1-\beta_j)\ \mathbf{P}_{i,j},\tag{19}$$

which can be written in a matrix form as

$$\begin{bmatrix}1-\frac{[m+1-i]}{[m+1]} & \frac{[m+1-i]}{[m+1]}\end{bmatrix} X \begin{bmatrix}\mathbf{P}_{i-1,j-1} & \mathbf{P}_{i-1,j}\\ \mathbf{P}_{i,j-1} & \mathbf{P}_{i,j}\end{bmatrix}\begin{bmatrix}1-\frac{[n+1-j]}{[n+1]}\\ \frac{[n+1-j]}{[n+1]}\end{bmatrix}.$$

De Casteljau algorithm can also be easily extended to evaluate points on a Bézier surface. Given the control points $\mathbf{P}_{i,j}\in\mathbb{R}^3, i=0,1,\cdots,m,\quad j=0,1,\cdots,n$, we have

$$\begin{cases} \mathbf{P_{i,j}^{0,0}}(u,v) \equiv \mathbf{P_{i,j}^{0,0}} \equiv \mathbf{P_{i,j}} i = 0,1,2\cdots,m; j = 0,1,2\cdots,n. \\ \mathbf{P_{i,j}^{r,r}}(u,v) = \left[\frac{m+\beta}{m}\left(\frac{m+\alpha}{m+\beta} - t\right) \quad \frac{m+\beta}{m}\left(t - \frac{\alpha}{m+\beta}\right)\right] \begin{bmatrix} \mathbf{P}_{i,j}^{r-1,r-1} & \mathbf{P}_{i,j+1}^{r-1,r-1} \\ \mathbf{P}_{i+1,j}^{r-1,r-1} & \mathbf{P}_{i+1,j+1}^{r-1,r-1} \end{bmatrix} \begin{bmatrix} \frac{n+\beta}{n}\left(\frac{n+\alpha}{n+\beta} - t\right) \\ \frac{n+\beta}{n}\left(t - \frac{\alpha}{n+\beta}\right) \end{bmatrix} \\ r = 1,\cdots,k,\ k = \min(m,n) \quad i = 0,1,2\cdots,m-r; \quad j = 0,1,\cdots,n-r, \quad \frac{\alpha}{n+\beta} \le t \le \frac{n+\alpha}{n+\beta}. \end{cases} \tag{20}$$

When $m = n$, one can directly use the algorithms above to get a point on the surface. When $m \neq n$, to get a point on the surface after k applications of formula (20), we use formula (15) for obtaining the intermediate point $\mathbf{P}_{i,j}^{k,k}$.

Note: We get classical Bézier surfaces for $(u,v) \in \left[\frac{\alpha}{n+\beta}, \frac{n+\alpha}{n+\beta}\right] \times \left[\frac{\alpha}{n+\beta}, \frac{n+\alpha}{n+\beta}\right]$ when we set the parameter $\alpha = \beta = 0$.

5. Concluding remarks

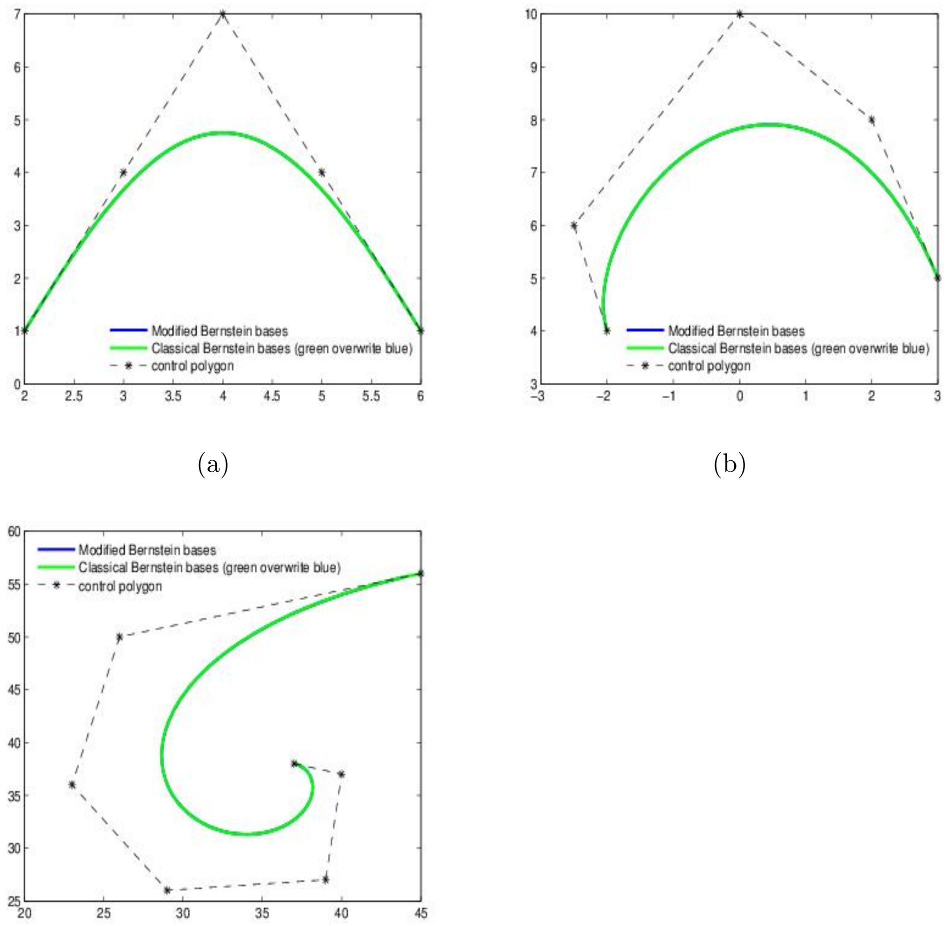
Bézier curves and surfaces are constructed with the help of modified Bernstein bases functions with shifted knots for $t \in \left[\frac{\alpha}{n+\beta}, \frac{n+\alpha}{n+\beta}\right]$. Introduction of parameters α and β enable us to shift Bernstein bases functions over subintervals of $[0,1]$. We get Bernstein functions on $[0,1]$, when we set the parameters $\alpha = \beta = 0$.

Simulation study is performed through MATLAB R2010a. It has been found out that Bézier curves that are generated over any subinterval of $[0,1]$ based on modified Bernstein bases functions which satisfy the properties of non negativity, partition of unity and end point interpolation property are similar to the Bézier curves that are generated based on classical Bernstein bases functions over the interval $[0,1]$. This result is shown in Figure 3 where Bézier curves based on modified Bernstein bases functions is shown by blue curve get overwrite by Bézier curves based on classical Bernstein bases functions shown by green curve.

6. MATLAB Code

```
m=[4,0];                      % value of alpha
l=[5,0];                      % value of beta

cpx=[37 40 39 29 23 26 45 ]   % x-coordinate of control points
cpy=[38 37 27 26 36 50 56 ]   % y-coordinate of control points
```



(a) (b)

(c)

Figure 3: Bézier curves based on modified Bernstein bases

`for shifloop=1:2`

`i=1;`
 $\alpha = m(shifloop); \beta = l(shifloop);$
`n=6;`
`d=1;`

`for` $y = \frac{\alpha}{n+\beta} : 0.01 : \frac{n+\alpha}{n+\beta}$

```
d=d+1;

end

bezierx=[1:d-1];
beziery=[1:d-1];
```

for $x = \frac{\alpha}{n+\beta} : 0.01 : \frac{n+\alpha}{n+\beta}$

```
bezierbernsteinx=0;bezierbernsteiny=0;

for k=0:n
```

$$bezierbernsteinx = bezierbernsteinx + \left[\begin{array}{c} n \\ k \end{array} \right] \left(x - \frac{\alpha}{n+\beta} \right)^k \left(\frac{n+\alpha}{n+\beta} - x \right)^{n-k} cpx(k+1);$$

$$bezierbernsteiny = bezierbernsteiny + \left[\begin{array}{c} n \\ k \end{array} \right] \left(x - \frac{\alpha}{n+\beta} \right)^k \left(\frac{n+\alpha}{n+\beta} - x \right)^{n-k} cpy(k+1);$$

```
end
```

$$bezierx(i) = bezierbernsteinx * \left(\frac{n+\beta}{n} \right)^n;$$
$$beziery(i) = bezierbernsteiny * \left(\frac{n+\beta}{n} \right)^n;$$

```
i=i+1;

end

if (shifloop==1)

c=plot(bezierx,beziery)
set(c,'Color','blue','LineWidth',2)

else

c=plot(bezierx,beziery,'g')
set(c,'Color','green','LineWidth',2)
```

```
end

hold on

end

plot(cpx,cpy,'--k*');
hleg1 = legend('Modified Bernstein bases','Classical Bernstein bases (green
overwrite blue)','control polygon','location','northwest');
set(hleg1,'Box','off')
hold on
```

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