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Some Results for Modular b-Metric Spaces and an Application to System of Linear Equations

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Abstract. In this paper, we define the modular b-metric space with some new notions and prove Banach fixed point theorem and its two generalizations for the new space. At the end of the paper, we give an application of Banach contraction principle to a system of linear equations.

Key Words and Phrases: fixed point, modular b-metric space, contraction principle.

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1. Introduction

Fixed point theory is a growing area in mathematics with significant properties. It has been used in many parts of science, for example, in engineering and computer science. Fixed point theory is the heart of mathematical analysis in the context of metric spaces. It has important applications in some fields, for example, in approximation theory.

Banach [3] established the Banach principle in 1922. Because of the simplicity of its structure, it has been used to solve some existence problems in many areas of mathematical analysis.

Czerwik [9] introduced the notion of b-metric space. His purpose was to overcome a problem about measurable functions so he extended the notion of metric space. For generalizations of contraction principle, in b-metric spaces, see [10]. In [12], it was proved that two theorems in cone b-metric spaces could be obtained from the result in the b-metric space. There are various studies [1, 13, 26, 29, 30, 31] in b-metric spaces.

Nakano [25] presented the concept of modular space. Modular spaces were studied by Musielak, Orlicz and other authors [21, 23, 24, 27, 28]. The modular

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metric space was introduced by Chistyakov who constructed a theory of this space [5]. He also obtained some results in [6, 7]. Cho et al. [8] proved the uniqueness of quasi-contractive mappings and the existence of fixed point, and Mongkolkeha et al. [22] gave some theorems on existence of fixed points related to contraction mappings in modular metric spaces. Dehghan et al. [11] gave an example concerning some results obtained in [22]. Several fixed point theorems on modular metric spaces were proved and a homotopy application was given in [14]. Ege and Alaca [15] introduced the modular S-metric spaces. In [16], modular ultrametric spaces were defined. Up to now many researchers had done essential works on modular metric spaces [2, 4, 17, 18, 20, 32].

In this paper, we give required definitions and theorems about modular and b-metric spaces. Next, we define a modular b-metric space and give definitions to prove Banach contraction principle in the new space. Finally, we give an application of this theory to the solution of linear equations.

2. Preliminaries

In this section, we deal with some notions required in modular and b-metric spaces.

Definition 1. *[27]. Let X be a real linear space. X is said to be a modular if a functional $p : X \longrightarrow [0, \infty]$ satisfies the following conditions:*

(A1) $p(0) = 0$;

(A2) If $a \in X$ and $p(ua) = 0$ for all numbers $u > 0$, then $a = 0$;

(A3) $p(-a) = p(a)$ for all $a \in X$;

(A4) $p(ua + vb) \le p(a) + p(b)$ for all $u, v \ge 0$ with $u + v = 1$ and $a, b \in X$.

Consider a set $X \neq \emptyset$ and $\lambda \in (0, \infty)$. In the rest of the paper, for all $\lambda > 0$ and $a, b \in X$, $\omega_\lambda(a, b) = \omega(\lambda, a, b)$ denotes the map $\omega : (0, \infty) \times X \times X \longrightarrow [0, \infty]$.

Definition 2. *[6]. For any set $X \neq \emptyset$, assume that the map $\omega : (0, \infty) \times X \times X \longrightarrow [0, \infty]$ satisfies the following conditions for all $a, b, c \in X$:*

(i) $\omega_\lambda(a, b) = 0$ for all $\lambda > 0 \iff a = b$;

(ii) $\omega_\lambda(a, b) = \omega_\lambda(b, a)$ for all $\lambda > 0$;

(iii) $\omega_{\lambda+\mu}(a, b) \le \omega_\lambda(a, c) + \omega_\mu(c, b)$ for all $\lambda, \mu > 0$.

Then we say that ω is a metric modular on X.

Definition 3. *[9]. Let $X \neq \emptyset$ be a set, $s \geq 1$ be a real number and a function $d : X \times X \longrightarrow \mathbb{R}_+$ satisfy the following conditions for all $u, v, w \in X$:*

1. $d(u, v) = 0 \Leftrightarrow u = v$,
2. $d(u, v) = d(v, u)$,
3. $d(u, w) \leq s[d(u, v) + d(v, w)]$.

Then we say that d is a b-metric and (X, d) is a b-metric space.

3. Modular b-metric spaces

Definition 4. *Let X be a non-empty set and let $s \geq 1$ be a real number. A map $\nu : (0, \infty) \times X \times X \longrightarrow [0, \infty]$ is called a modular b-metric, if the following statements hold for all $x, y, z \in X$,*

(i) $\nu_\lambda(x, y) = 0$ *for all* $\lambda > 0 \;\Leftrightarrow\; x = y$;

(ii) $\nu_\lambda(x, y) = \nu_\lambda(y, x)$ *for all* $\lambda > 0$;

(iii) $\nu_{\lambda+\mu}(x, y) \leq s[\nu_\lambda(x, z) + \nu_\mu(z, y)]$ *for all* $\lambda, \mu > 0$.

Then we say that (X, ν) is a modular b-metric space.

The modular b-metric space could be seen as a generalization of the modular metric space.

Example 1. *Consider the space*

$$l_p = \left\{ (x_n) \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}, 0 < p < 1,$$

$\lambda \in (0, \infty)$ *and* $\nu_\lambda(x, y) = \dfrac{d(x, y)}{\lambda}$ *such that*

$$d(x, y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{\frac{1}{p}}, \quad x = x_n, y = y_n \in l_p.$$

It could be easily seen that (X, ν) is a modular b-metric space.

Indeed, it is obvious that $d(x, z) \leq 2^{\frac{1}{p}}[d(x, y) + d(y, z)]$ is a b-metric [19].

$$\begin{aligned} \nu_{\lambda+\mu}(x, z) = \frac{d(x, z)}{\lambda + \mu} &\leq 2^{\frac{1}{p}}[\frac{d(x, y)}{\lambda + \mu} + \frac{d(y, z)}{\lambda + \mu}] \\ &\leq 2^{\frac{1}{p}}[\frac{d(x, y)}{\lambda} + \frac{d(y, z)}{\mu}] \\ &= 2^{\frac{1}{p}}[\nu_\lambda(x, y) + \nu_\mu(y, z)]. \end{aligned}$$

Definition 5. *Let ν be a modular b-metric on a set X. For $x, y \in X$, the binary relation $\overset{\nu}{\sim}$ on X defined by*

$$x \overset{\nu}{\sim} y \;\Leftrightarrow\; \lim_{\lambda \to \infty} \nu_\lambda(x, y) = 0$$

is an equivalence relation. A modular set is defined by

$$X_\nu = \{y \in X : y \overset{\nu}{\sim} x\}.$$

We define state the set

$$X_\nu^* = \{x \in X : \exists \lambda = \lambda(x) > 0 \;\; \text{such that} \;\; \nu_\lambda(x, x_0) < \infty\} \quad (x_0 \in X)$$

as well.

Now let's present definitions of ν-Cauchy, ν-convergent sequences and ν-complete space.

Definition 6. *Let (X, ν) be a modular b-metric space.*

- *A sequence $(x_n)_{n \in \mathbb{N}}$ in X_ν^* is called ν-convergent to $x \in X_\nu^*$ if $\nu_\lambda(x_n, x) \to 0$, as $n \to \infty$ for all $\lambda > 0$.*
- *A sequence $(x_n)_{n \in \mathbb{N}} \subset X_\nu^*$ is said to be ν-Cauchy if and only if for all $\epsilon > 0$ there exists $n(\epsilon) \in \mathbb{N}$ such that for each $n, m \geq n(\epsilon)$ and $\lambda > 0$ we have $\nu_\lambda(x_n, x_m) < \epsilon$.*
- *A modular b-metric space X_ν^* is ν-complete if each ν-Cauchy sequence in X_ν^* is ν-convergent and its limit is in X_ν^*.*

Definition 7. *Let ν be a modular b-metric on a set X and $T : X_\nu^* \longrightarrow X_\nu^*$ be an arbitrary map. If for every $x, y \in X_\nu^*$ and all $\lambda > 0$ there exists $0 < k < 1$ such that*

$$\nu_\lambda(Tx, Ty) \leq k\nu_\lambda(x, y),$$

then the map T is said to be a ν-contraction.

We now prove the Banach contraction principle in a modular b-metric space.

Theorem 1. *Let X_ν^* be ν-complete and $T : X_\nu^* \longrightarrow X_\nu^*$ be a ν-contraction with two restrictions $k \in [0, 1)$ and $sk < 1$ where $s \geq 1$. Assume that there exists $x = x(\lambda) \in X_\nu^*$ such that $\nu_\lambda(x, Tx) < \infty$. Then there is an element $\bar{x} \in X_\nu^*$ such that $x_n \to \bar{x}$ and $\bar{x}$ is a unique fixed point of T.*

Proof. Let $x_0 \in X_\nu^*$ and a sequence $(x_n)_{n\in\mathbb{N}}$ in X_ν^* be defined by

$$x_n = Tx_{n-1} = T^n x_0 \quad (n \in \mathbb{N}).$$

We get

$$\nu_\lambda(T^2x_0, T^2x_1) \leq k\nu_\lambda(Tx_0, Tx_1) \leq k^2\nu_\lambda(x_0, x_1)$$

because T is a ν-contraction. If this procedure is iterated, we obtain

$$\nu_\lambda(T^n x_0, T^n x_1) \leq k^n \nu_\lambda(x_0, x_1).$$

Let's show that $(x_n)_{n\in\mathbb{N}}$ is ν-Cauchy in X_ν^*.

$$\begin{aligned}\nu_\lambda(x_n, x_m) \leq & s\nu_{\frac{\lambda}{m-n}}(x_n, x_{n+1}) + s^2\nu_{\frac{\lambda}{m-n}}(x_{n+1}, x_{n+2}) + \ldots + s^{m-n}\nu_{\frac{\lambda}{m-n}}(x_{m-1}, x_m)\\ \leq & sk^n\nu_{\frac{\lambda}{m-n}}(x_0, x_1) + s^2k^{n+1}\nu_{\frac{\lambda}{m-n}}(x_0, x_1) + \ldots + s^{m-n}k^{m-1}\nu_{\frac{\lambda}{m-n}}(x_0, x_1)\\ = & \nu_{\frac{\lambda}{m-n}}(x_0, x_1)sk^n[1 + sk + (sk)^2 + \ldots + (sk)^{m-n-1}], \end{aligned} \tag{1}$$

where $m, n > 0$ with $m > n$. Letting $n, m \to \infty$ in (1), we have

$$\lim_{n,m\to\infty} \nu_\lambda(x_n, x_m) = 0.$$

Thus, $(x_n)_{n\in\mathbb{N}}$ is a ν-Cauchy sequence in X_ν^*.

By the ν-completeness of X_ν^*, we conclude that $(x_n)_{n\in\mathbb{N}}$ is ν-convergent to $\bar{x} \in X_\nu^*$. Since

$$\begin{aligned}\nu_\lambda(T\bar{x}, \bar{x}) &\leq s[\nu_{\frac{\lambda}{2}}(T\bar{x}, x_n) + \nu_{\frac{\lambda}{2}}(x_n, \bar{x})]\\ &\leq s[k\nu_{\frac{\lambda}{2}}(\bar{x}, x_{n-1}) + \nu_{\frac{\lambda}{2}}(x_n, \bar{x})]\\ &\overset{n\to\infty}{=} 0,\end{aligned}$$

$T\bar{x} = \bar{x}$ and so $\bar{x}$ is the fixed point of T.

We finally prove that $\bar{x}$ is the unique fixed point of T. Assume that y is another fixed point of T. Then we have $Ty = y$. Since

$$\nu_\lambda(\bar{x}, y) = \nu_\lambda(T\bar{x}, Ty) \leq k\nu_\lambda(\bar{x}, y),$$

we obtain

$$(1-k)\nu_\lambda(\bar{x}, y) \leq 0 \quad \Rightarrow \quad \nu_\lambda(\bar{x}, y) = 0 \quad \Rightarrow \quad \bar{x} = y.$$

◄

In the rest of the section, we give two generalizations of Theorem 1.

Theorem 2. *Let X^*_ν be ν-complete. Suppose that $T : X^*_\nu \longrightarrow X^*_\nu$ is a mapping with two restrictions $s\alpha \in \left(0, \frac{1}{2}\right)$ and $\alpha \in \left[0, \frac{1}{2}\right)$ such that*

$$\nu_\lambda(Tx, Ty) \le \alpha[\nu_\lambda(x, Tx) + \nu_\lambda(y, Ty)] \tag{2}$$

for each $x, y \in X^*_\nu$. Assume that there exists $x = x(\lambda) \in X^*_\nu$ such that $\nu_\lambda(x, Tx) < \infty$. Then there exists $\overline{x} \in X^*_\nu$ such that $x_n \to \overline{x}$ and $\overline{x}$ is the unique fixed point of T.

Proof. Let $x_0 \in X^*_\nu$. For a sequence $\{x_n\}_{n=1}^\infty$ in X^*_ν such that $x_n = Tx_{n-1} = T^n x_0, n = 1, 2, \cdots$, by (2) we have

$$\begin{aligned}\nu_\lambda(x_n, x_{n+1}) &= \nu_\lambda(Tx_{n-1}, Tx_n)\\ &\le \alpha[\nu_\lambda(x_{n-1}, x_n) + \nu_\lambda(x_n, x_{n+1})]\end{aligned}$$

and

$$\nu_\lambda(x_n, x_{n+1}) \le \frac{\alpha}{1-\alpha}\nu_\lambda(x_{n-1}, x_n).$$

The following inequality could be obtained by the method used above:

$$\nu_\lambda(x_n, x_{n+1}) \le \left(\frac{\alpha}{1-\alpha}\right)^n \nu_\lambda(x_0, x_1). \tag{3}$$

Noting that $\alpha \in \left[0, \frac{1}{2}\right)$, we get $\frac{\alpha}{1-\alpha} \in [0, 1)$. Therefore, T is a ν-contraction. From (3), we conclude that

$$\nu_\lambda(x_n, x_m) < s\left(\frac{\alpha}{1-\alpha}\right)^n \frac{1 - \left(\frac{s\alpha}{1-\alpha}\right)^{m-n}}{1 - \frac{s\alpha}{1-\alpha}} \nu_\lambda(x_0, x_1).$$

Letting $n, m \to \infty$, we obtain that $\{x_n\}_{n=1}^\infty$ is ν-Cauchy and so ν-convergent. Let $\{x_n\}_{n=1}^\infty$ be convergent to $\overline{x} \in X^*_\nu$. Then we obtain

$$\begin{aligned}\nu_\lambda(\overline{x}, T\overline{x}) &\le s[\nu_\lambda(\overline{x}, x_n) + \nu_\lambda(x_n, T\overline{x})]\\ &\le s\nu_\lambda(\overline{x}, x_n) + s\alpha[\nu_\lambda(x_{n-1}, x_n) + \nu_\lambda(\overline{x}, T\overline{x})]\\ &\le \frac{s}{1-s\alpha}\nu_\lambda(\overline{x}, x_n) + \frac{s\alpha}{1-s\alpha}\nu_\lambda(x_{n-1}, x_n).\end{aligned}$$

Moreover, (3) implies that

$$\nu_\lambda(\overline{x}, T\overline{x}) \le \frac{s}{1-s\alpha}\nu_\lambda(\overline{x}, x_n) + \frac{s\alpha}{1-s\alpha}\left(\frac{\alpha}{1-\alpha}\right)^{n-1} \nu_\lambda(x_0, x_1).$$

Letting $n \to \infty$,

$$\lim_{n\to\infty} \nu_\lambda(\overline{x}, T\overline{x}) = 0,$$

we have $\overline{x} = T\overline{x}$.

Let's prove the uniqueness of $\overline{x}$. If $v \neq \overline{x}$ is another fixed point of T, then we get

$$\begin{aligned} 0 < \nu_\lambda(\overline{x}, v) = \nu_\lambda(T\overline{x}, Tv) &\leq \alpha[\nu_\lambda(\overline{x}, T\overline{x}) + \nu_\lambda(v, Tv)] \\ &= \alpha[\nu_\lambda(\overline{x}, \overline{x}) + \nu_\lambda(v, v)] \\ &= 0, \end{aligned}$$

which is a contradiction. Thus there exist no other fixed points of T. ◀

Theorem 3. *Let X_ν^* be ν-complete and let $T : X_\nu^* \longrightarrow X_\nu^*$ be a mapping with $sk \in \left[0, \frac{1}{2}\right)$ for which there exists $k \in \left[0, \frac{1}{2}\right)$ such that*

$$\nu_\lambda(Tx, Ty) \leq k[\nu_\lambda(x, Ty) + \nu_\lambda(y, Tx)] \tag{4}$$

for all $x, y \in X_\nu^*$. Assume that there exists $x = x(\lambda) \in X_\nu^*$ such that $\nu_\lambda(x, Tx) < \infty$. Then there exists $\overline{x} \in X_\nu^*$ such that $x_n \to \overline{x}$ and $\overline{x}$ is the unique fixed point of T.

Proof. Let $x_0 \in X_\nu^*$ and consider a sequence $\{x_n\}_{n=1}^\infty$ in X_ν^* defined as $x_n = Tx_{n-1} = T^n x_0, n = 1, 2, \cdots$. Since we get

$$\begin{aligned} \nu_\lambda(x_n, x_{n+1}) = \nu_\lambda(Tx_{n-1}, Tx_n) &\leq k[\nu_\lambda(x_{n-1}, Tx_n) + \nu_\lambda(x_n, Tx_{n-1})] \\ &= k[\nu_\lambda(x_{n-1}, x_{n+1}) + \nu_\lambda(x_n, x_n)] \\ &\leq sk[\nu_\lambda(x_{n-1}, x_n) + \nu_\lambda(x_n, x_{n+1})], \end{aligned}$$

from (4) we obtain

$$\nu_\lambda(x_n, x_{n+1}) \leq \frac{sk}{1 - sk} \nu_\lambda(x_{n-1}, x_n).$$

Thus, T is a ν-contraction because $sk \in \left[0, \frac{1}{2}\right)$ and so $\frac{sk}{1-sk} \in [0, 1)$. Proceeding the same way as in the proofs of Theorems 1 and 2, it is easy to see that $\{x_n\}_{n=1}^\infty$ is a ν-Cauchy and hence a convergent sequence.

Let $\overline{x} \in X_\nu^*$ be the point of convergence of $\{x_n\}_{n=1}^\infty$. We will show that $\overline{x}$ is a fixed point of T. Since

$$\begin{aligned} \nu_\lambda(\overline{x}, T\overline{x}) &\leq s[\nu_\lambda(\overline{x}, x_{n+1}) + \nu_\lambda(x_{n+1}, T\overline{x})] \\ &= s\nu_\lambda(\overline{x}, x_{n+1}) + s\nu_\lambda(Tx_n, T\overline{x}) \\ &\leq s\nu_\lambda(\overline{x}, x_{n+1}) + sk[\nu_\lambda(\overline{x}, Tx_n) + \nu_\lambda(x_n, T\overline{x})], \end{aligned}$$

we have

$$\nu_\lambda(\overline{x}, T\overline{x}) \leq s\nu_\lambda(\overline{x}, x_{n+1}) + sk\nu_\lambda(\overline{x}, x_{n+1}) + sk\nu_\lambda(x_n, T\overline{x}).$$

Letting $n \to \infty$,

$$\nu_\lambda(\overline{x}, T\overline{x}) \leq sk\nu_\lambda(\overline{x}, T\overline{x}). \tag{5}$$

If $\nu_\lambda(\overline{x}, T\overline{x}) \neq 0$, then the inequality (5) is false. So we get $\overline{x} = T\overline{x}$.

In order to show that $\overline{x}$ is the unique fixed point of T, we assume that y is another fixed point of T, i.e. $Ty = y$. Then we get

$$\nu_\lambda(\overline{x}, y) = \nu_\lambda(T\overline{x}, Ty) \leq k[\nu_\lambda(\overline{x}, Ty) + \nu_\lambda(y, T\overline{x})],$$

and so $\nu_\lambda(\overline{x}, y) \leq 2k\nu_\lambda(\overline{x}, y)$. The last inequality implies that $y = \overline{x}$. ◄

4. An application to a system of linear equations

In this section, our aim is to find solutions of a system of linear equations via the Theorem 1. For this reason, we prove the following theorem.

Theorem 4. *Let $X = \mathbb{R}^n$ be a modular b-metric space with modular b-metric*

$$\nu_\lambda(x, y) = \frac{d(x, y)}{\lambda},$$

where $x, y \in X$ and

$$d(x, y) = \sum_{i=1}^{n} |x_i - y_i|.$$

If

$$\sum_{i=1}^{n} |\alpha_{ij}| \leq \alpha < 1 \quad \textit{for all } j = 1, 2, \ldots, n,$$

then the linear system

$$\begin{cases} u_{11}x_1 + u_{12}x_2 + \ldots + u_{1n}x_n = v_1 \\ u_{21}x_1 + u_{22}x_2 + \ldots + u_{2n}x_n = v_2 \\ \vdots \\ u_{n1}x_1 + u_{n2}x_2 + \ldots + u_{nn}x_n = v_n \end{cases} \tag{6}$$

of n linear equations in n unknowns has a unique solution.

Proof. Since $X = \mathbb{R}^n$ with ν_λ modular b-metric is ν-complete, we need to prove that $T : X \to X$ defined by

$$T(x) = Ax + v,$$

where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$ and

$$A = \begin{pmatrix} u_{11} & u_{12} & \ldots & u_{1n} \\ u_{21} & u_{22} & \ldots & u_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ u_{n1} & u_{n2} & \ldots & u_{nn} \end{pmatrix}$$

is a ν-contraction. Since

$$\begin{aligned} \nu_\lambda(Tx, Ty) = \frac{1}{\lambda}\sum_{i=1}^{n}\left|\sum_{j=1}^{n}\alpha_{ij}(x_j - y_j)\right| &\leq \frac{1}{\lambda}\sum_{i=1}^{n}\sum_{j=1}^{n}|\alpha_{ij}||x_j - y_j| \\ &= \frac{1}{\lambda}\sum_{j=1}^{n}\sum_{i=1}^{n}|\alpha_{ij}||x_j - y_j| \\ &\leq \frac{\alpha}{\lambda}\sum_{j=1}^{n}|x_j - y_j| \\ &= \alpha\frac{d(x,y)}{\lambda} \\ &= \alpha\nu_\lambda(x,y), \end{aligned}$$

we conclude that T is a ν-contraction mapping. By Theorem 1, the linear equation system (6) has a unique solution. ◀

5. Conclusion

In recent years, searching for new fixed point theorems in various metric spaces has been a significant study. The aim of researchers in the area of fixed point theory is to obtain interesting and useful applications of fixed point theorems. For this reason, we give some results on modular b-metric spaces and give an application of these results to a system of linear equations. We hope that these results will develop the fixed point theory.

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